

Regular holonomic $\mathcal{D}[[\hbar]]$ -modules

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Abstract

We describe the category of regular holonomic modules over the ring $\mathcal{D}[[\hbar]]$ of linear differential operators with a formal parameter \hbar . In particular, we establish the Riemann-Hilbert correspondence and discuss the additional t -structure related to \hbar -torsion.

Introduction

On a complex manifold X , we will be interested in the study of holonomic modules over the ring $\mathcal{D}_X[[\hbar]]$ of differential operators with a formal parameter \hbar . Such modules naturally appear when studying deformation quantization modules (DQ-modules) along a smooth Lagrangian submanifold of a complex symplectic manifold (see [11, Chapter 7]).

In this paper, after recalling the tools from loc. cit. that we shall use, we explain some basic notions of $\mathcal{D}_X[[\hbar]]$ -modules theory. For example, it follows easily from general results on modules over $\mathbb{C}[[\hbar]]$ -algebras that given two holonomic $\mathcal{D}_X[[\hbar]]$ -modules \mathcal{M} and \mathcal{N} , the complex $R\mathcal{H}om_{\mathcal{D}_X[[\hbar]]}(\mathcal{M}, \mathcal{N})$ is constructible over $\mathbb{C}[[\hbar]]$ and the microsupport of the solution complex $R\mathcal{H}om_{\mathcal{D}_X[[\hbar]]}(\mathcal{M}, \mathcal{O}_X[[\hbar]])$ coincides with the characteristic variety of \mathcal{M} .

Then we establish our main result, the Riemann-Hilbert correspondence for regular holonomic $\mathcal{D}_X[[\hbar]]$ -modules, an \hbar -variant of Kashiwara's classical theorem. In other words, we show that the solution functor with values in $\mathcal{O}_X[[\hbar]]$ induces an equivalence between the derived category of regular holonomic $\mathcal{D}_X[[\hbar]]$ -modules and that of constructible sheaves over $\mathbb{C}[[\hbar]]$. A quasi-inverse is obtained by constructing the “sheaf” of holomorphic functions with temperate growth and a formal parameter \hbar in the subanalytic site. This needs some care since the literature on this subject is written in the framework of sheaves over a field and does not immediately apply to the ring $\mathbb{C}[[\hbar]]$.

We also discuss the t -structure related to \hbar -torsion. Indeed, as we work over the ring $\mathbb{C}[[\hbar]]$ and not over a field, the derived category of holonomic $\mathcal{D}_X[[\hbar]]$ -modules (or, equivalently, that of constructible sheaves over $\mathbb{C}[[\hbar]]$) has an additional t -structure related to \hbar -torsion. We will show how the duality functor interchanges it with the natural t -structure.

Finally, we describe some natural links between the ring $\mathcal{D}_X[[\hbar]]$ and deformation quantization algebras, as mentioned above.

Notations and conventions

We shall mainly follow the notations of [10]. In particular, if \mathcal{C} is an abelian category, we denote by $D(\mathcal{C})$ the derived category of \mathcal{C} and by $D^*(\mathcal{C})$ ($*$ = +, −, b) the full triangulated subcategory consisting of objects with bounded from below (resp. bounded from above, resp. bounded) cohomology.

For a sheaf of rings \mathcal{R} on a topological space, or more generally a site, we denote by $\text{Mod}(\mathcal{R})$ the category of left \mathcal{R} -modules and we write $D^*(\mathcal{R})$ instead of $D^*(\text{Mod}(\mathcal{R}))$ ($*$ = \emptyset , +, −, b). We denote by $\text{Mod}_{\text{coh}}(\mathcal{R})$ the full abelian subcategory of $\text{Mod}(\mathcal{R})$ of coherent objects, and by $D_{\text{coh}}^b(\mathcal{R})$ the full triangulated subcategory of $D^b(\mathcal{R})$ of objects with coherent cohomology groups.

If R is a ring (a sheaf of rings over a point), we write for short $D_f^b(R)$ instead of $D_{\text{coh}}^b(R)$.

1 Formal deformations (after [11])

We review here some definitions and results from [11] that we shall use in this paper.

Modules over $\mathbb{Z}[[\hbar]]$ -algebras

One says that a $\mathbb{Z}[[\hbar]]$ -module \mathcal{M} has no \hbar -torsion if $\hbar: \mathcal{M} \rightarrow \mathcal{M}$ is injective and one says that \mathcal{M} is \hbar -complete if $\mathcal{M} \rightarrow \varprojlim^n \mathcal{M}/\hbar^n \mathcal{M}$ is an isomorphism.

Let \mathcal{R} be a $\mathbb{Z}[[\hbar]]$ -algebra, and assume that \mathcal{R} has no \hbar -torsion. One sets

$$\mathcal{R}^{\text{loc}} := \mathbb{Z}[[\hbar, \hbar^{-1}]] \otimes_{\mathbb{Z}[[\hbar]]} \mathcal{R}, \quad \mathcal{R}_0 := \mathcal{R}/\hbar \mathcal{R},$$

and considers the functors

$$\begin{aligned} (\bullet)^{\text{loc}}: \text{Mod}(\mathcal{R}) &\rightarrow \text{Mod}(\mathcal{R}^{\text{loc}}), \quad \mathcal{M} \mapsto \mathcal{M}^{\text{loc}} := \mathcal{R}^{\text{loc}} \otimes_{\mathcal{R}} \mathcal{M}, \\ \text{gr}_{\hbar}: \text{D}(\mathcal{R}) &\rightarrow \text{D}(\mathcal{R}_0), \quad \mathcal{M} \mapsto \text{gr}_{\hbar}(\mathcal{M}) := \mathcal{R}_0^{\text{L}} \otimes_{\mathcal{R}} \mathcal{M}. \end{aligned}$$

Note that $(\bullet)^{\text{loc}}$ is exact and that for $\mathcal{M}, \mathcal{N} \in \text{D}^{\text{b}}(\mathcal{R})$ and $\mathcal{P} \in \text{D}^{\text{b}}(\mathcal{R}^{\text{op}})$ one has isomorphisms:

$$(1.1) \quad \text{gr}_{\hbar}(\mathcal{P} \otimes_{\mathcal{R}}^{\text{L}} \mathcal{M}) \simeq \text{gr}_{\hbar} \mathcal{P} \otimes_{\mathcal{R}_0}^{\text{L}} \text{gr}_{\hbar} \mathcal{M},$$

$$(1.2) \quad \text{gr}_{\hbar}(\text{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{N})) \simeq \text{R}\mathcal{H}om_{\mathcal{R}_0}(\text{gr}_{\hbar}(\mathcal{M}), \text{gr}_{\hbar}(\mathcal{N})).$$

Cohomologically \hbar -complete sheaves

Definition 1.1. One says that an object \mathcal{M} of $\text{D}(\mathcal{R})$ is cohomologically \hbar -complete if $\text{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{R}^{\text{loc}}, \mathcal{M}) = 0$.

Hence, the full subcategory of cohomologically \hbar -complete objects is triangulated. In fact, it is the right orthogonal to the full subcategory $\text{D}(\mathcal{R}^{\text{loc}})$ of $\text{D}(\mathcal{R})$.

Remark that $\mathcal{M} \in \text{D}(\mathcal{R})$ is cohomologically \hbar -complete if and only if its image in $\text{D}(\mathbb{Z}_X[\hbar])$ is cohomologically \hbar -complete.

Proposition 1.2. *Let $\mathcal{M} \in \text{D}(\mathcal{R})$. Then \mathcal{M} is cohomologically \hbar -complete if and only if*

$$\varinjlim_{U \ni x} \text{Ext}_{\mathbb{Z}[\hbar]}^j(\mathbb{Z}[\hbar, \hbar^{-1}], H^i(U; \mathcal{M})) = 0,$$

for any $x \in X$, any integer $i \in \mathbb{Z}$ and any $j = 0, 1$. Here, U ranges over an open neighborhood system of x .

Corollary 1.3. *Let $\mathcal{M} \in \text{Mod}(\mathcal{R})$. Assume that \mathcal{M} has no \hbar -torsion, is \hbar -complete and there exists a base \mathfrak{B} of open subsets such that $H^i(U; \mathcal{M}) = 0$ for any $i > 0$ and any $U \in \mathfrak{B}$. Then \mathcal{M} is cohomologically \hbar -complete.*

The functor gr_{\hbar} is conservative on the category of cohomologically \hbar -complete objects:

Proposition 1.4. *Let $\mathcal{M} \in \text{D}(\mathcal{R})$ be a cohomologically \hbar -complete object. If $\text{gr}_{\hbar}(\mathcal{M}) = 0$, then $\mathcal{M} = 0$.*

Proposition 1.5. *Assume that $\mathcal{M} \in \mathbf{D}(\mathcal{R})$ is cohomologically \hbar -complete. Then $\mathbf{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{N}, \mathcal{M}) \in \mathbf{D}(\mathbb{Z}_X[\hbar])$ is cohomologically \hbar -complete for any $\mathcal{N} \in \mathbf{D}(\mathcal{R})$.*

Proposition 1.6. *Let $f: X \rightarrow Y$ be a continuous map, and $\mathcal{M} \in \mathbf{D}(\mathbb{Z}_X[\hbar])$. If \mathcal{M} is cohomologically \hbar -complete, then so is $\mathbf{R}f_*\mathcal{M}$.*

Reductions to $\hbar = 0$

Now we assume that X is a Hausdorff locally compact topological space.

By a basis \mathfrak{B} of compact subsets of X , we mean a family of compact subsets such that for any $x \in X$ and any open neighborhood U of x , there exists $K \in \mathfrak{B}$ such that $x \in \text{Int}(K) \subset U$.

Let \mathcal{A} be a $\mathbb{Z}[\hbar]$ -algebra, and recall that we set $\mathcal{A}_0 = \mathcal{A}/\hbar\mathcal{A}$. Consider the following conditions:

- (i) \mathcal{A} has no \hbar -torsion and is \hbar -complete,
- (ii) \mathcal{A}_0 is a left Noetherian ring,
- (iii) there exists a basis \mathfrak{B} of compact subsets of X and a prestack $U \mapsto \text{Mod}_{\text{gd}}(\mathcal{A}_0|_U)$ (U open in X) such that
 - (a) for any $K \in \mathfrak{B}$ and an open subset U such that $K \subset U$, there exists $K' \in \mathfrak{B}$ such that $K \subset \text{Int}(K') \subset K' \subset U$,
 - (b) $U \mapsto \text{Mod}_{\text{gd}}(\mathcal{A}_0|_U)$ is a full subprestack of $U \mapsto \text{Mod}_{\text{coh}}(\mathcal{A}_0|_U)$,
 - (c) for any $K \in \mathfrak{B}$, any open set U containing K , any $\mathcal{M} \in \text{Mod}_{\text{gd}}(\mathcal{A}_0|_U)$ and any $j > 0$, one has $H^j(K; \mathcal{M}) = 0$,
 - (d) for an open subset U and $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{A}_0|_U)$, if $\mathcal{M}|_V$ belongs to $\text{Mod}_{\text{gd}}(\mathcal{A}_0|_V)$ for any relatively compact open subset V of U , then \mathcal{M} belongs to $\text{Mod}_{\text{gd}}(\mathcal{A}_0|_U)$,
 - (e) for any U open in X , $\text{Mod}_{\text{gd}}(\mathcal{A}_0|_U)$ is stable by subobjects, quotients and extensions in $\text{Mod}_{\text{coh}}(\mathcal{A}_0|_U)$,
 - (f) for any $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{A}_0|_U)$, there exists an open covering $U = \bigcup_i U_i$ such that $\mathcal{M}|_{U_i} \in \text{Mod}_{\text{gd}}(\mathcal{A}_0|_{U_i})$,
 - (g) $\mathcal{A}_0 \in \text{Mod}_{\text{gd}}(\mathcal{A}_0)$,
- (iii)' there exists a basis \mathfrak{B} of open subsets of X such that for any $U \in \mathfrak{B}$, any $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{A}_0|_U)$ and any $j > 0$, one has $H^j(U; \mathcal{M}) = 0$.

We will suppose that \mathcal{A} and \mathcal{A}_0 satisfy either Assumption 1.7 or Assumption 1.8.

Assumption 1.7. \mathcal{A} and \mathcal{A}_0 satisfy conditions (i), (ii) and (iii) above.

Assumption 1.8. \mathcal{A} and \mathcal{A}_0 satisfy conditions (i), (ii) and (iii)' above.

Theorem 1.9. (i) \mathcal{A} is a left Noetherian ring.

(ii) Any coherent \mathcal{A} -module \mathcal{M} is \hbar -complete.

(iii) Let $\mathcal{M} \in \mathbf{D}_{\text{coh}}^{\text{b}}(\mathcal{A})$. Then \mathcal{M} is cohomologically \hbar -complete.

Corollary 1.10. The functor $\text{gr}_{\hbar}: \mathbf{D}_{\text{coh}}^{\text{b}}(\mathcal{A}) \rightarrow \mathbf{D}_{\text{coh}}^{\text{b}}(\mathcal{A}_0)$ is conservative.

Theorem 1.11. Let $\mathcal{M} \in \mathbf{D}^+(\mathcal{A})$ and assume:

(a) \mathcal{M} is cohomologically \hbar -complete,

(b) $\text{gr}_{\hbar}(\mathcal{M}) \in \mathbf{D}_{\text{coh}}^+(\mathcal{A}_0)$.

Then, $\mathcal{M} \in \mathbf{D}_{\text{coh}}^+(\mathcal{A})$ and for all $i \in \mathbb{Z}$ we have the isomorphism

$$H^i(\mathcal{M}) \xrightarrow{\sim} \varprojlim_n H^i(\mathcal{A}/\hbar^n \mathcal{A} \overset{\text{L}}{\otimes}_{\mathcal{A}} \mathcal{M}).$$

Theorem 1.12. Assume that $\mathcal{A}_0^{\text{op}} = \mathcal{A}^{\text{op}}/\hbar \mathcal{A}^{\text{op}}$ is a Noetherian ring and the flabby dimension of X is finite. Let \mathcal{M} be an \mathcal{A} -module. Assume the following conditions:

(a) \mathcal{M} has no \hbar -torsion,

(b) \mathcal{M} is cohomologically \hbar -complete,

(c) $\mathcal{M}/\hbar \mathcal{M}$ is a flat \mathcal{A}_0 -module.

Then \mathcal{M} is a flat \mathcal{A} -module.

If moreover $\mathcal{M}/\hbar \mathcal{M}$ is a faithfully flat \mathcal{A}_0 -module, then \mathcal{M} is a faithfully flat \mathcal{A} -module.

Theorem 1.13. Let $d \in \mathbb{N}$. Assume that \mathcal{A}_0 is d -syzygic, i.e., that any coherent \mathcal{A}_0 -module locally admits a projective resolution of length $\leq d$ by free \mathcal{A}_0 -modules of finite rank. Then

- (a) \mathcal{A} is $(d+1)$ -syzygic.
- (b) Let \mathcal{M}^\bullet be a complex of \mathcal{A} -modules concentrated in degrees $[a, b]$ and with coherent cohomology groups. Then, locally there exists a quasi-isomorphism $\mathcal{L}^\bullet \rightarrow \mathcal{M}^\bullet$ where \mathcal{L}^\bullet is a complex of free \mathcal{A} -modules of finite rank concentrated in degrees $[a-d-1, b]$.

Proposition 1.14. *Let $\mathcal{M} \in \mathbf{D}_{\text{coh}}^b(\mathcal{A})$ and let $a \in \mathbb{Z}$. The conditions below are equivalent:*

- (i) $H^a(\text{gr}_{\hbar}(\mathcal{M})) \simeq 0$,
- (ii) $H^a(\mathcal{M}) \simeq 0$ and $H^{a+1}(\mathcal{M})$ has no \hbar -torsion.

Cohomologically \hbar -complete sheaves on real manifolds

Let now X be a real analytic manifold. Recall from [7] that the microsupport of $F \in \mathbf{D}^b(\mathbb{Z}_X)$ is a closed involutive subset of the cotangent bundle T^*X denoted by $\text{SS}(F)$. The microsupport is additive on $\mathbf{D}^b(\mathbb{Z}_X)$ (cf Definition 3.3 (ii) below). Considering the distinguished triangle $F \xrightarrow{\hbar} F \rightarrow \text{gr}_{\hbar} F \xrightarrow{+1}$, one gets the estimate

$$(1.3) \quad \text{SS}(\text{gr}_{\hbar}(F)) \subset \text{SS}(F).$$

Using Proposition 1.4 and 1.6, one easily proves:

Proposition 1.15. *Let $F \in \mathbf{D}^b(\mathbb{Z}_X[[\hbar]])$ and assume that F is cohomologically \hbar -complete. Then*

$$(1.4) \quad \text{SS}(F) = \text{SS}(\text{gr}_{\hbar}(F)).$$

For \mathbb{K} a commutative unital Noetherian ring, one denotes by $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$ the full subcategory of $\text{Mod}(\mathbb{K}_X)$ consisting of \mathbb{R} -constructible sheaves and by $\mathbf{D}_{\mathbb{R}\text{-c}}^b(\mathbb{K}_X)$ the full triangulated subcategory of $\mathbf{D}^b(\mathbb{K}_X)$ consisting of objects with \mathbb{R} -constructible cohomology. In this paper, we shall mainly be interested with the case where \mathbb{K} is either \mathbb{C} or the ring of formal power series in an indeterminate \hbar , that we denote by

$$\mathbb{C}^{\hbar} := \mathbb{C}[[\hbar]].$$

By Proposition 1.2 one has

Proposition 1.16. *Let $F \in D_{\mathbb{R}-c}^b(\mathbb{C}_X^h)$. Then F is cohomologically \hbar -complete.*

Corollary 1.17. *The functor $\mathrm{gr}_{\hbar}: D_{\mathbb{R}-c}^b(\mathbb{C}_X^h) \rightarrow D_{\mathbb{R}-c}^b(\mathbb{C}_X)$ is conservative.*

Corollary 1.18. *For $F \in D_{\mathbb{R}-c}^b(\mathbb{C}_X^h)$, one has the equality*

$$\mathrm{SS}(\mathrm{gr}_{\hbar}(F)) = \mathrm{SS}(F).$$

Proposition 1.19. *For $F \in D_{\mathbb{R}-c}^b(\mathbb{C}_X^h)$ and $i \in \mathbb{Z}$ one has $\mathrm{supp} H^i(F) \subset \mathrm{supp} H^i(\mathrm{gr}_{\hbar} F)$. In particular if $H^i(\mathrm{gr}_{\hbar} F) = 0$ then $H^i(F) = 0$.*

Proof. We apply Proposition 1.14 to F_x for any $x \in X$.

Q.E.D.

2 Formal extension

Let X be a topological space, or more generally a site, and let \mathcal{R}_0 be a sheaf of rings on X . In this section, we let

$$\mathcal{R} := \mathcal{R}_0[[\hbar]] = \prod_{n \geq 0} \mathcal{R}_0 \hbar^n$$

be the formal extension of \mathcal{R}_0 , whose sections on an open subset U are formal series $r = \sum_{n=0}^{\infty} r_j \hbar^n$, with $r_j \in \Gamma(U; \mathcal{R}_0)$. Consider the associated functor

$$(2.1) \quad (\bullet)^{\hbar}: \mathrm{Mod}(\mathcal{R}_0) \rightarrow \mathrm{Mod}(\mathcal{R}),$$

$$\mathcal{N} \mapsto \mathcal{N}[[\hbar]] = \varprojlim_n (\mathcal{R}_n \otimes_{\mathcal{R}_0} \mathcal{N}),$$

where $\mathcal{R}_n := \mathcal{R}/\hbar^{n+1}\mathcal{R}$ is regarded as an $(\mathcal{R}, \mathcal{R}_0)$ -bimodule. Since \mathcal{R}_n is free of finite rank over \mathcal{R}_0 , the functor $(\bullet)^{\hbar}$ is left exact. We denote by $(\bullet)^{\mathrm{R}\hbar}$ its right derived functor.

Proposition 2.1. *For $\mathcal{N} \in D^b(\mathcal{R}_0)$ one has*

$$\mathcal{N}^{\mathrm{R}\hbar} \simeq \mathrm{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}^{\mathrm{loc}}/\hbar\mathcal{R}, \mathcal{N}),$$

where $\mathcal{R}^{\mathrm{loc}}/\hbar\mathcal{R}$ is regarded as an $(\mathcal{R}_0, \mathcal{R})$ -bimodule.

Proof. It is enough to prove that for $\mathcal{N} \in \mathrm{Mod}(\mathcal{R}_0)$ one has

$$\mathcal{N}^{\hbar} \simeq \mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}^{\mathrm{loc}}/\hbar\mathcal{R}, \mathcal{N}).$$

Let $\mathcal{R}_n^* = \mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}_n, \mathcal{R}_0)$, regarded as an $(\mathcal{R}_0, \mathcal{R})$ -bimodule. Then

$$\mathcal{N}^{\hbar} = \varprojlim_n (\mathcal{R}_n \otimes_{\mathcal{R}_0} \mathcal{N}) \simeq \mathcal{H}om_{\mathcal{R}_0}(\varinjlim_n \mathcal{R}_n^*, \mathcal{N}).$$

Since

$$\mathcal{R}^{\text{loc}}/\hbar\mathcal{R} \simeq \varinjlim_n (\hbar^{-n}\mathcal{R}/\hbar\mathcal{R}),$$

it is enough to prove that there is an isomorphism of $(\mathcal{R}_0, \mathcal{R})$ -bimodules

$$\mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}_n, \mathcal{R}_0) \simeq \hbar^{-n}\mathcal{R}/\hbar\mathcal{R}.$$

Recalling that $\mathcal{R}_n = \mathcal{R}/\hbar^{n+1}\mathcal{R}$, this follows from the pairing

$$(\mathcal{R}/\hbar^{n+1}\mathcal{R}) \otimes_{\mathcal{R}_0} (\hbar^{-n}\mathcal{R}/\hbar\mathcal{R}) \rightarrow \mathcal{R}_0, \quad f \otimes g \mapsto \text{Res}_{\hbar=0}(fg d\hbar/\hbar).$$

Q.E.D.

Note that the isomorphism of $(\mathcal{R}, \mathcal{R}_0)$ -bimodules

$$\mathcal{R} \simeq (\mathcal{R}_0)^{\hbar} = \mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}, \mathcal{R}_0)$$

induces a natural morphism

$$(2.2) \quad \mathcal{R} \overset{\text{L}}{\otimes}_{\mathcal{R}_0} \mathcal{N} \rightarrow \mathcal{N}^{\text{R}\hbar}, \quad \text{for } \mathcal{N} \in \text{D}^b(\mathcal{R}_0).$$

Proposition 2.2. *For $\mathcal{N} \in \text{D}^b(\mathcal{R}_0)$, its formal extension $\mathcal{N}^{\text{R}\hbar}$ is cohomologically \hbar -complete.*

Proof. The statement follows from $(\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}) \overset{\text{L}}{\otimes}_{\mathcal{R}} \mathcal{R}^{\text{loc}} \simeq 0$ and from the isomorphism

$$\text{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{R}^{\text{loc}}, \mathcal{N}^{\text{R}\hbar}) \simeq \text{R}\mathcal{H}om_{\mathcal{R}_0}((\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}) \overset{\text{L}}{\otimes}_{\mathcal{R}} \mathcal{R}^{\text{loc}}, \mathcal{N}).$$

Q.E.D.

Lemma 2.3. *Assume that \mathcal{R}_0 is an \mathcal{S}_0 -algebra, for \mathcal{S}_0 a commutative sheaf of rings, and let $\mathcal{S} = \mathcal{S}_0[[\hbar]]$. For $\mathcal{M}, \mathcal{N} \in \text{D}^b(\mathcal{R}_0)$ we have an isomorphism in $\text{D}^b(\mathcal{S})$*

$$\text{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{M}, \mathcal{N})^{\text{R}\hbar} \simeq \text{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{M}, \mathcal{N}^{\text{R}\hbar}).$$

Proof. Noticing that $\mathcal{R}^{\text{loc}}/\hbar\mathcal{R} \simeq \mathcal{R}_0 \otimes_{\mathcal{S}_0} (\mathcal{S}^{\text{loc}}/\hbar\mathcal{S})$ as $(\mathcal{R}_0, \mathcal{S})$ -bimodules, one has

$$\begin{aligned} \mathrm{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{M}, \mathcal{N})^{\mathrm{Rh}} &= \mathrm{R}\mathcal{H}om_{\mathcal{S}_0}(\mathcal{S}^{\text{loc}}/\hbar\mathcal{S}, \mathrm{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{M}, \mathcal{N})) \\ &\simeq \mathrm{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}, \mathrm{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{M}, \mathcal{N})) \\ &\simeq \mathrm{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{M}, \mathrm{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}, \mathcal{N})) \\ &= \mathrm{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{M}, \mathcal{N}^{\mathrm{Rh}}). \end{aligned}$$

Q.E.D.

Lemma 2.4. *Let $f: Y \rightarrow X$ be a morphism of sites, and assume that $(f^{-1}\mathcal{R}_0)^{\hbar} \simeq f^{-1}\mathcal{R}$. Then the functors $\mathrm{R}f_*$ and $(\bullet)^{\mathrm{Rh}}$ commute, that is, for $\mathcal{P} \in \mathrm{D}^b(f^{-1}\mathcal{R}_0)$ we have $(\mathrm{R}f_*\mathcal{P})^{\mathrm{Rh}} \simeq \mathrm{R}f_*(\mathcal{P}^{\mathrm{Rh}})$ in $\mathrm{D}^b(\mathcal{R})$.*

Proof. One has the isomorphism

$$\begin{aligned} \mathrm{R}f_*(\mathcal{P}^{\mathrm{Rh}}) &= \mathrm{R}f_*\mathrm{R}\mathcal{H}om_{f^{-1}\mathcal{R}_0}(f^{-1}(\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}), \mathcal{P}) \\ &\simeq \mathrm{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}, \mathrm{R}f_*\mathcal{P}) \\ &= \mathrm{R}f_*(\mathcal{P}^{\mathrm{Rh}}). \end{aligned}$$

Q.E.D.

Proposition 2.5. *Let \mathcal{T} be either a basis of open subsets of the site X or, assuming that X is a locally compact topological space, a basis of compact subsets. Denote by $J_{\mathcal{T}}$ the full subcategory of $\mathrm{Mod}(\mathcal{R}_0)$ consisting of \mathcal{T} -acyclic objects, i.e., sheaves \mathcal{N} for which $H^k(S; \mathcal{N}) = 0$ for all $k > 0$ and all $S \in \mathcal{T}$. Then $J_{\mathcal{T}}$ is injective with respect to the functor $(\bullet)^{\hbar}$. In particular, for $\mathcal{N} \in J_{\mathcal{T}}$, we have $\mathcal{N}^{\hbar} \simeq \mathcal{N}^{\mathrm{Rh}}$.*

Proof. (i) Since injective sheaves are \mathcal{T} -acyclic, $J_{\mathcal{T}}$ is cogenerating.

(ii) Consider an exact sequence $0 \rightarrow \mathcal{N}' \rightarrow \mathcal{N} \rightarrow \mathcal{N}'' \rightarrow 0$ in $\mathrm{Mod}(\mathcal{R}_0)$. Clearly, if both \mathcal{N}' and \mathcal{N} belong to $J_{\mathcal{T}}$, then so does \mathcal{N}'' .

(iii) Consider an exact sequence as in (ii) and assume that $\mathcal{N}' \in J_{\mathcal{T}}$. We have to prove that $0 \rightarrow \mathcal{N}'^{\hbar} \rightarrow \mathcal{N}^{\hbar} \rightarrow \mathcal{N}''^{\hbar} \rightarrow 0$ is exact. Since $(\bullet)^{\hbar}$ is left exact, it is enough to prove that $\mathcal{N}^{\hbar} \rightarrow \mathcal{N}''^{\hbar}$ is surjective. Noticing that $\mathcal{N}^{\hbar} \simeq \prod_{\mathbb{N}} \mathcal{N}$ as \mathcal{R}_0 -modules, it is enough to prove that $\prod_{\mathbb{N}} \mathcal{N} \rightarrow \prod_{\mathbb{N}} \mathcal{N}''$ is surjective.

(iii)-(a) Assume that \mathcal{T} is a basis of open subsets. Any open subset $U \subset X$ has a cover $\{U_i\}_{i \in I}$ by elements $U_i \in \mathcal{T}$. For any $i \in I$, the morphism

$\mathcal{N}(U_i) \rightarrow \mathcal{N}''(U_i)$ is surjective. The result follows taking the product over \mathbb{N} .

(iii)-(b) Assume that \mathcal{T} is a basis of compact subsets. For any $K \in \mathcal{T}$, the morphism $\mathcal{N}(K) \rightarrow \mathcal{N}''(K)$ is surjective. Hence, there exists a basis \mathcal{V} of open subsets such that for any $x \in X$ and any $V \ni x$ in \mathcal{V} , there exists $V' \in \mathcal{V}$ with $x \in V' \subset V$ and the image of $\mathcal{N}(V') \rightarrow \mathcal{N}''(V')$ contains the image of $\mathcal{N}''(V)$ in $\mathcal{N}''(V')$. The result follows as in (iii)-(a) taking the product over \mathbb{N} . Q.E.D.

Corollary 2.6. *The following sheaves are acyclic for the functor $(\bullet)^h$:*

- (i) *\mathbb{R} -constructible sheaves of \mathbb{C} -vector spaces on a real analytic manifold X (see [7, §8.4]),*
- (ii) *coherent modules over the ring \mathcal{O}_X of holomorphic functions on a complex analytic manifold X ,*
- (iii) *coherent modules over the ring \mathcal{D}_X of linear differential operators on a complex analytic manifold X .*

Proof. The statements follow by applying Proposition 2.5 for the following choices of \mathcal{T} .

- (i) Let F be an \mathbb{R} -constructible sheaf. Then for any $x \in X$ one has $F_x \xleftarrow{\sim} \mathrm{R}\Gamma(U_x; F)$ for U_x in a fundamental system of open neighborhoods of x . Take for \mathcal{T} the union of these fundamental systems.
- (ii) Take for \mathcal{T} the family of open Stein subsets.
- (iii) Let \mathcal{M} be a coherent \mathcal{D}_X -module. The problem being local, we may assume that \mathcal{M} is endowed with a good filtration. Then take for \mathcal{T} the family of compact Stein subsets. Q.E.D.

Example 2.7. Let $X = \mathbb{R}$, $\mathcal{R}_0 = \mathbb{C}_X$, $Z = \{1/n : n = 1, 2, \dots\} \cup \{0\}$ and $U = X \setminus Z$. One has the isomorphisms $(\mathbb{C}^h)_X \simeq (\mathbb{C}_X)^h \simeq (\mathbb{C}_X)^{\mathrm{Rh}}$ and $(\mathbb{C}^h)_U \simeq (\mathbb{C}_U)^h$. Considering the exact sequences

$$\begin{aligned} 0 &\rightarrow (\mathbb{C}^h)_U \rightarrow (\mathbb{C}^h)_X \rightarrow (\mathbb{C}^h)_Z \rightarrow 0, \\ 0 &\rightarrow (\mathbb{C}_U)^h \rightarrow (\mathbb{C}_X)^h \rightarrow (\mathbb{C}_Z)^h \rightarrow H^1(\mathbb{C}_U)^{\mathrm{Rh}} \rightarrow 0, \end{aligned}$$

we get $H^1(\mathbb{C}_U)^{\mathrm{Rh}} \simeq (\mathbb{C}_Z)^h / (\mathbb{C}^h)_Z$, whose stalk at the origin does not vanish. Hence \mathbb{C}_U is not acyclic for the functor $(\bullet)^h$.

Assume now that

$$\mathcal{A}_0 = \mathcal{R}_0 \quad \text{and} \quad \mathcal{A} = \mathcal{R}_0[[\hbar]]$$

satisfy either Assumption 1.7 or Assumption 1.8 (where condition (i) is clear) and that \mathcal{A}_0 is syzygic. Note that by Proposition 2.5 one has $\mathcal{A} \simeq (\mathcal{A}_0)^{\text{R}\hbar}$.

Proposition 2.8. *For $\mathcal{N} \in \text{D}_{\text{coh}}^{\text{b}}(\mathcal{A}_0)$:*

- (i) *there is an isomorphism $\mathcal{N}^{\text{R}\hbar} \xrightarrow{\sim} \mathcal{A} \overset{\text{L}}{\otimes}_{\mathcal{A}_0} \mathcal{N}$ induced by (2.2),*
- (ii) *there is an isomorphism $\text{gr}_{\hbar}(\mathcal{N}^{\hbar}) \simeq \mathcal{N}$.*

Proof. Since \mathcal{A}_0 is syzygic, we may locally represent \mathcal{N} by a bounded complex \mathcal{L}^{\bullet} of free \mathcal{A}_0 -modules of finite rank. Then (i) is obvious. As for (ii), both complexes are isomorphic to the mapping cone of $\hbar: (\mathcal{L}^{\bullet})^{\hbar} \rightarrow (\mathcal{L}^{\bullet})^{\hbar}$.
Q.E.D.

In particular, the functor $(\cdot)^{\hbar}$ is exact on $\text{Mod}_{\text{coh}}(\mathcal{A}_0)$ and preserves coherence. One thus get a functor

$$(\cdot)^{\text{R}\hbar}: \text{D}_{\text{coh}}^{\text{b}}(\mathcal{A}_0) \rightarrow \text{D}_{\text{coh}}^{\text{b}}(\mathcal{A}).$$

The subanalytic site

The subanalytic site associated to an analytic manifold X has been introduced and studied in [9, Chapter 7] (see also [13] for a detailed and systematic study as well as for complementary results). Denote by Op_X the category of open subsets of X , the morphisms being the inclusion morphisms, and by $\text{Op}_{X_{\text{sa}}}$ the full subcategory consisting of relatively compact subanalytic open subsets of X . The site X_{sa} is the presite $\text{Op}_{X_{\text{sa}}}$ endowed with the Grothendieck topology for which the coverings are those admitting a finite subcover. One calls X_{sa} the subanalytic site associated to X . Denote by $\rho: X \rightarrow X_{\text{sa}}$ the natural morphism of sites. Recall that the inverse image functors ρ^{-1} , besides the usual right adjoint given by the direct image functor ρ_* , admits a left adjoint denoted $\rho_!$. Consider the diagram

$$\begin{array}{ccc} \text{D}^{\text{b}}(\mathbb{C}_X) & \overset{\text{R}\rho_*}{\underset{\rho^{-1}}{\rightleftarrows}} & \text{D}^{\text{b}}(\mathbb{C}_{X_{\text{sa}}}) \\ \downarrow (\cdot)^{\text{R}\hbar} & & \downarrow (\cdot)^{\text{R}\hbar} \\ \text{D}^{\text{b}}(\mathbb{C}_X^{\hbar}) & \overset{\text{R}\rho_*}{\underset{\rho^{-1}}{\rightleftarrows}} & \text{D}^{\text{b}}(\mathbb{C}_{X_{\text{sa}}}^{\hbar}). \end{array}$$

Lemma 2.9. (i) *The functors ρ^{-1} and $(\bullet)^{\text{Rh}}$ commute, that is, for $G \in \text{D}^b(\mathbb{C}_{X_{\text{sa}}})$ we have $(\rho^{-1}G)^{\text{Rh}} \simeq \rho^{-1}(G^{\text{Rh}})$ in $\text{D}^b(\mathbb{C}_X^h)$.*

(ii) *The functors $R\rho_*$ and $(\bullet)^{\text{Rh}}$ commute, that is, for $F \in \text{D}^b(\mathbb{C}_X)$ we have $(R\rho_*F)^{\text{Rh}} \simeq R\rho_*(F^{\text{Rh}})$ in $\text{D}^b(\mathbb{C}_{X_{\text{sa}}}^h)$.*

Proof. (i) Since it admits a left adjoint, the functor ρ^{-1} commutes with projective limits. It follows that for $G \in \text{Mod}(\mathbb{C}_{X_{\text{sa}}})$ one has an isomorphism

$$\rho^{-1}(G^h) \rightarrow (\rho^{-1}G)^h.$$

To conclude, it remains to show that $(\rho^{-1}(\bullet))^{\text{Rh}}$ is the derived functor of $(\rho^{-1}(\bullet))^h$. Recall that an object G of $\text{Mod}(\mathbb{C}_{X_{\text{sa}}})$ is quasi-injective if the functor $\text{Hom}_{\mathbb{C}_{X_{\text{sa}}}}(\bullet, G)$ is exact on the category $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. By a result of [13], if $G \in \text{Mod}(\mathbb{C}_{X_{\text{sa}}})$ is quasi-injective, then $\rho^{-1}G$ is soft. Hence, $\rho^{-1}G$ is injective for the functor $(\bullet)^h$ by Proposition 2.5.

(ii) By (i) we can apply Lemma 2.4.

Q.E.D.

3 $\mathcal{D}[[\hbar]]$ -modules and propagation

Let now X be a complex analytic manifold of complex dimension d_X . As usual, denote by \mathbb{C}_X the constant sheaf with stalk \mathbb{C} , by \mathcal{O}_X the structure sheaf and by \mathcal{D}_X the ring of linear differential operators on X . We will use the notations

$$\begin{aligned} \text{D}' : \text{D}^b(\mathbb{C}_X)^{\text{op}} &\rightarrow \text{D}^b(\mathbb{C}_X), & F &\mapsto \text{R}\mathcal{H}om_{\mathbb{C}_X}(F, \mathbb{C}_X), \\ \mathbb{D} : \text{D}_{\text{coh}}^b(\mathcal{D}_X)^{\text{op}} &\rightarrow \text{D}_{\text{coh}}^b(\mathcal{D}_X), & \mathcal{M} &\mapsto \text{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X], \\ \text{Sol} : \text{D}_{\text{coh}}^b(\mathcal{D}_X)^{\text{op}} &\rightarrow \text{D}^b(\mathbb{C}_X), & \mathcal{M} &\mapsto \text{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X), \\ \text{DR} : \text{D}_{\text{coh}}^b(\mathcal{D}_X) &\rightarrow \text{D}^b(\mathbb{C}_X), & \mathcal{M} &\mapsto \text{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}), \end{aligned}$$

where Ω_X denotes the line bundle of holomorphic forms of maximal degree and $\Omega_X^{\otimes -1}$ the dual bundle.

As shown in Corollary 2.6, the sheaves \mathbb{C}_X , \mathcal{O}_X and \mathcal{D}_X are all acyclic for the functor $(\bullet)^h$. We will be interested in the formal extensions

$$\mathbb{C}_X^h = \mathbb{C}_X[[\hbar]], \quad \mathcal{O}_X^h = \mathcal{O}_X[[\hbar]], \quad \mathcal{D}_X^h = \mathcal{D}_X[[\hbar]].$$

In the sequel, we shall treat left \mathcal{D}_X^h -modules, but all results apply to right modules since the categories $\text{Mod}(\mathcal{D}_X^h)$ and $\text{Mod}(\mathcal{D}_X^{h,\text{op}})$ are equivalent.

Proposition 3.1. *The \mathbb{C}^h -algebras \mathcal{D}_X^h and $\mathcal{D}_X^{h,\text{op}}$ satisfy Assumptions 1.7.*

Proof. Assumption 1.7 hold for $\mathcal{A} = \mathcal{D}_X^h$, $\mathcal{A}_0 = \mathcal{D}_X$, $\text{Mod}_{\text{gd}}(\mathcal{A}_0|_U)$ the category of good \mathcal{D}_U -modules (see [5]) and for \mathfrak{B} the family of Stein compact subsets of X . Q.E.D.

In particular, by Theorem 1.11 one has that \mathcal{D}_X^h is right and left Noetherian (and thus coherent). Moreover, by Theorem 1.13 any object of $\text{D}_{\text{coh}}^b(\mathcal{D}_X^h)$ can be locally represented by a bounded complex of free \mathcal{D}_X^h -modules of finite rank.

We will use the notations

$$\begin{aligned} \text{D}'_h: \text{D}^b(\mathbb{C}_X^h)^{\text{op}} &\rightarrow \text{D}^b(\mathbb{C}_X^h), & F &\mapsto \text{R}\mathcal{H}om_{\mathbb{C}_X^h}(F, \mathbb{C}_X^h), \\ \mathbb{D}_h: \text{D}_{\text{coh}}^b(\mathcal{D}_X^h)^{\text{op}} &\rightarrow \text{D}_{\text{coh}}^b(\mathcal{D}_X^h), & \mathcal{M} &\mapsto \text{R}\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{M}, \mathcal{D}_X^h \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X], \\ \text{Sol}_h: \text{D}_{\text{coh}}^b(\mathcal{D}_X^h)^{\text{op}} &\rightarrow \text{D}^b(\mathbb{C}^h), & \mathcal{M} &\mapsto \text{R}\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{M}, \mathcal{O}_X^h), \\ \text{DR}_h: \text{D}_{\text{coh}}^b(\mathcal{D}_X^h) &\rightarrow \text{D}^b(\mathbb{C}^h), & \mathcal{M} &\mapsto \text{R}\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{O}_X^h, \mathcal{M}). \end{aligned}$$

By Proposition 2.8 and Lemma 2.3, for $\mathcal{N} \in \text{D}_{\text{coh}}^b(\mathcal{D}_X)$ one has

$$(3.1) \quad \mathcal{N}^{\text{Rh}} \simeq \mathcal{D}_X^h \overset{\text{L}}{\otimes}_{\mathcal{D}_X} \mathcal{N},$$

$$(3.2) \quad \text{gr}_h(\mathcal{N}^{\text{Rh}}) \simeq \mathcal{N},$$

$$(3.3) \quad \text{Sol}_h(\mathcal{N}^{\text{Rh}}) \simeq \text{Sol}(\mathcal{N})^{\text{Rh}}.$$

Definition 3.2. For $\mathcal{M} \in \text{Mod}(\mathcal{D}_X^h)$, denote by $\mathcal{M}_{h\text{-tor}}$ its submodule consisting of sections locally annihilated by some power of \hbar and set $\mathcal{M}_{h\text{-tf}} = \mathcal{M} / \mathcal{M}_{h\text{-tor}}$. We say that $\mathcal{M} \in \text{Mod}(\mathcal{D}_X^h)$ is an \hbar -torsion module if $\mathcal{M}_{h\text{-tor}} \xrightarrow{\sim} \mathcal{M}$ and that \mathcal{M} has no \hbar -torsion (or is \hbar -torsion free) if $\mathcal{M} \xrightarrow{\sim} \mathcal{M}_{h\text{-tf}}$.

Denote by ${}_n\mathcal{M}$ the kernel of $\hbar^{n+1}: \mathcal{M} \rightarrow \mathcal{M}$. Then $\mathcal{M}_{h\text{-tor}}$ is the sheaf associated with the increasing union of the ${}_n\mathcal{M}$'s. Hence, if \mathcal{M} is coherent, the increasing family $\{{}_n\mathcal{M}\}_n$ is locally stationary and $\mathcal{M}_{h\text{-tor}}$ as well as $\mathcal{M}_{h\text{-tf}}$ are coherent.

Characteristic variety

Recall the following definition

Definition 3.3. (i) For \mathcal{C} an abelian category, a function $c: \text{Ob}(\mathcal{C}) \rightarrow \text{Set}$ is called additive if $c(M) = c(M') \cup c(M'')$ for any short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$.
(ii) For \mathcal{T} a triangulated category, a function $c: \text{Ob}(\mathcal{T}) \rightarrow \text{Set}$ is called additive if $c(M) = c(M[1])$ and $c(M) \subset c(M') \cup c(M'')$ for any distinguished triangle $M' \rightarrow M \rightarrow M'' \xrightarrow{+1}$.

Note that an additive function c on \mathcal{C} naturally extend to the derived category $\text{D}(\mathcal{C})$ by setting $c(M) = \bigcup_i c(H^i(M))$.

For \mathcal{N} a coherent \mathcal{D}_X -module, denote by $\text{char}(\mathcal{N})$ its characteristic variety, a closed involutive subvariety of the cotangent bundle T^*X . The characteristic variety is additive on $\text{Mod}_{\text{coh}}(\mathcal{D}_X)$. For $\mathcal{N} \in \text{D}_{\text{coh}}^b(\mathcal{D}_X)$ one sets $\text{char}(\mathcal{N}) = \bigcup_i \text{char}(H^i(\mathcal{N}))$.

Definition 3.4. The characteristic variety of $\mathcal{M} \in \text{D}_{\text{coh}}^b(\mathcal{D}_X^h)$ is defined by

$$\text{char}_h(\mathcal{M}) = \text{char}(\text{gr}_h(\mathcal{M})).$$

To $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^h)$ one associates the coherent \mathcal{D}_X -modules

$$(3.4) \quad {}_0\mathcal{M} = \text{Ker}(\hbar: \mathcal{M} \rightarrow \mathcal{M}) = H^{-1}(\text{gr}_h \mathcal{M}),$$

$$(3.5) \quad \mathcal{M}_0 = \text{Coker}(\hbar: \mathcal{M} \rightarrow \mathcal{M}) = H^0(\text{gr}_h \mathcal{M}).$$

Lemma 3.5. For $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^h)$ an \hbar -torsion module, one has

$$\text{char}_h(\mathcal{M}) = \text{char}(\mathcal{M}_0) = \text{char}({}_0\mathcal{M}).$$

Proof. By definition, $\text{char}_h(\mathcal{M}) = \text{char}(\mathcal{M}_0) \cup \text{char}({}_0\mathcal{M})$. It is thus enough to prove the equality $\text{char}(\mathcal{M}_0) = \text{char}({}_0\mathcal{M})$.

Since the statement is local we may assume that $\hbar^N \mathcal{M} = 0$ for some $N \in \mathbb{N}$. We proceed by induction on N .

For $N = 1$ we have $\mathcal{M} \simeq \mathcal{M}_0 \simeq {}_0\mathcal{M}$, and the statement is obvious.

Assume that the statement has been proved for $N - 1$. The short exact sequence

$$(3.6) \quad 0 \rightarrow \hbar \mathcal{M} \rightarrow \mathcal{M} \rightarrow \mathcal{M}_0 \rightarrow 0$$

induces the distinguished triangle

$$\text{gr}_h \hbar \mathcal{M} \rightarrow \text{gr}_h \mathcal{M} \rightarrow \text{gr}_h \mathcal{M}_0 \xrightarrow{+1}.$$

Noticing that $\mathcal{M}_0 \simeq (\mathcal{M}_0)_0 \simeq {}_0(\mathcal{M}_0)$, the associated long exact cohomology sequence gives

$$0 \rightarrow {}_0(\hbar\mathcal{M}) \rightarrow {}_0\mathcal{M} \rightarrow \mathcal{M}_0 \rightarrow (\hbar\mathcal{M})_0 \rightarrow 0.$$

By inductive hypothesis we have $\text{char}({}_0(\hbar\mathcal{M})) = \text{char}((\hbar\mathcal{M})_0)$, and we deduce $\text{char}(\mathcal{M}_0) = \text{char}(\mathcal{M}_0)$ by additivity of $\text{char}(\bullet)$. Q.E.D.

Proposition 3.6. (i) For $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar})$ one has

$$\text{char}_{\hbar}(\mathcal{M}) = \text{char}(\mathcal{M}_0).$$

(ii) The characteristic variety $\text{char}_{\hbar}(\bullet)$ is additive both on $\text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar})$ and on $\text{D}^b(\mathcal{D}_X^{\hbar})$.

Proof. (i) As $\text{char}(\text{gr}_{\hbar}\mathcal{M}) = \text{char}(\mathcal{M}_0) \cup \text{char}({}_0\mathcal{M})$, it is enough to prove the inclusion

$$(3.7) \quad \text{char}({}_0\mathcal{M}) \subset \text{char}(\mathcal{M}_0).$$

Consider the short exact sequence $0 \rightarrow \mathcal{M}_{\hbar\text{-tor}} \rightarrow \mathcal{M} \rightarrow \mathcal{M}_{\hbar\text{-tf}} \rightarrow 0$. Since $\mathcal{M}_{\hbar\text{-tf}}$ has no \hbar -torsion, ${}_0(\mathcal{M}_{\hbar\text{-tf}}) = 0$. The associated long exact cohomology sequence thus gives

$${}_0(\mathcal{M}_{\hbar\text{-tor}}) \simeq {}_0\mathcal{M}, \quad 0 \rightarrow (\mathcal{M}_{\hbar\text{-tor}})_0 \rightarrow \mathcal{M}_0 \rightarrow (\mathcal{M}_{\hbar\text{-tf}})_0 \rightarrow 0.$$

We deduce

$$\text{char}({}_0\mathcal{M}) = \text{char}({}_0(\mathcal{M}_{\hbar\text{-tor}})) = \text{char}((\mathcal{M}_{\hbar\text{-tor}})_0) \subset \text{char}(\mathcal{M}_0),$$

where the second equality follows from Lemma 3.5.

(ii) It is enough to prove the additivity on $\text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar})$, i.e. the equality

$$\text{char}_{\hbar}(\mathcal{M}) = \text{char}_{\hbar}(\mathcal{M}') \cup \text{char}_{\hbar}(\mathcal{M}'')$$

for $0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0$ a short exact sequence of coherent \mathcal{D}_X^{\hbar} -modules.

The associated distinguished triangle $\text{gr}_{\hbar}\mathcal{M}' \rightarrow \text{gr}_{\hbar}\mathcal{M} \rightarrow \text{gr}_{\hbar}\mathcal{M}'' \xrightarrow{+1}$ induces the long exact cohomology sequence

$${}_0(\mathcal{M}'') \rightarrow (\mathcal{M}')_0 \rightarrow \mathcal{M}_0 \rightarrow (\mathcal{M}'')_0 \rightarrow 0.$$

By additivity of $\text{char}(\cdot)$, the exactness of this sequence at the first, second and third term from the right, respectively, gives:

$$\begin{aligned}\text{char}_{\hbar}(\mathcal{M}'') &\subset \text{char}_{\hbar}(\mathcal{M}), \\ \text{char}_{\hbar}(\mathcal{M}) &\subset \text{char}_{\hbar}(\mathcal{M}') \cup \text{char}_{\hbar}(\mathcal{M}''), \\ \text{char}_{\hbar}(\mathcal{M}') &\subset \text{char}({}_0(\mathcal{M}'')) \cup \text{char}_{\hbar}(\mathcal{M}).\end{aligned}$$

Finally, note that $\text{char}({}_0(\mathcal{M}'')) \subset \text{char}_{\hbar}(\mathcal{M}'') \subset \text{char}_{\hbar}(\mathcal{M})$. Q.E.D.

Remark 3.7. In view of Proposition 3.6 (i), in order to define the characteristic variety of a coherent \mathcal{D}_X^{\hbar} -module \mathcal{M} one could avoid derived categories considering $\text{char}(\mathcal{M}_0)$ instead of $\text{char}(\text{gr}_{\hbar} \mathcal{M})$. It is then natural to ask if these definitions are still compatible for $\mathcal{M} \in \text{D}_{\text{coh}}^{\text{b}}(\mathcal{D}_X^{\hbar})$, i.e. to ask if the following equality holds

$$\bigcup_i \text{char}(H^i(\text{gr}_{\hbar} \mathcal{M})) = \bigcup_i \text{char}((H^i \mathcal{M})_0).$$

Let us prove it. By additivity of $\text{char}(\cdot)$, the short exact sequence

$$0 \rightarrow (H^i \mathcal{M})_0 \rightarrow H^i(\text{gr}_{\hbar} \mathcal{M}) \rightarrow {}_0(H^{i+1} \mathcal{M}) \rightarrow 0$$

from [11, Lemma 1.4.2] induces the estimates

$$\begin{aligned}\text{char}((H^i \mathcal{M})_0) &\subset \text{char}(H^i(\text{gr}_{\hbar} \mathcal{M})), \\ \text{char}(H^i(\text{gr}_{\hbar} \mathcal{M})) &= \text{char}((H^i \mathcal{M})_0) \cup \text{char}({}_0(H^{i+1} \mathcal{M})).\end{aligned}$$

One concludes by noticing that (3.7) gives

$$\text{char}({}_0(H^{i+1} \mathcal{M})) \subset \text{char}((H^{i+1} \mathcal{M})_0).$$

Proposition 3.8. *Let $\mathcal{M} \in \text{Mod}(\mathcal{D}_X^{\hbar})$ be an \hbar -torsion module. Then \mathcal{M} is coherent as a \mathcal{D}_X^{\hbar} -module if and only if it is coherent as a \mathcal{D}_X -module, and in this case one has $\text{char}_{\hbar}(\mathcal{M}) = \text{char}(\mathcal{M})$.*

Proof. As in the proof of Lemma 3.5 we assume that $\hbar^N \mathcal{M} = 0$ for some $N \in \mathbb{N}$. Since coherence is preserved by extension and since the characteristic varieties of \mathcal{D}_X^{\hbar} -modules and \mathcal{D}_X -modules are additive, we can argue by induction on N using the exact sequence (3.6). We are thus reduced to the case $N = 1$, where $\mathcal{M} = \mathcal{M}_0$ and the statement becomes obvious. Q.E.D.

It follows from (3.2) that

Proposition 3.9. *For $\mathcal{N} \in \text{D}_{\text{coh}}^{\text{b}}(\mathcal{D}_X)$ one has $\text{char}_{\hbar}(\mathcal{N}^{\hbar}) = \text{char}(\mathcal{N})$.*

Holonomic modules

Recall that a coherent \mathcal{D}_X -module (or an object of the derived category) is called holonomic if its characteristic variety is isotropic. We refer e.g. to [5, Chapter 5] for the notion of regularity.

Definition 3.10. We say that $\mathcal{M} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_X^h)$ is holonomic, or regular holonomic, if so is $\text{gr}_h(\mathcal{M})$. We denote by $\mathbf{D}_{\text{hol}}^b(\mathcal{D}_X^h)$ the full triangulated subcategory of $\mathbf{D}_{\text{coh}}^b(\mathcal{D}_X^h)$ of holonomic objects and by $\mathbf{D}_{\text{rh}}^b(\mathcal{D}_X^h)$ the full triangulated subcategory of regular holonomic objects.

Note that a coherent \mathcal{D}_X^h -module is holonomic if and only if its characteristic variety is isotropic.

Example 3.11. Let \mathcal{N} be a regular holonomic \mathcal{D}_X -module. Then

- (i) \mathcal{N} itself, considered as a \mathcal{D}_X^h -module, is regular holonomic, as follows from the isomorphism $\text{gr}_h \mathcal{N} \simeq \mathcal{N} \oplus \mathcal{N}[1]$;
- (ii) \mathcal{N}^h is a regular holonomic \mathcal{D}_X^h -module, as follows from the isomorphism $\text{gr}_h \mathcal{N}^h \simeq \mathcal{N}$. In particular, \mathcal{O}_X^h is regular holonomic.

Propagation

Denote by $\mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^h)$ the full triangulated subcategory of $\mathbf{D}^b(\mathbb{C}_X^h)$ consisting of objects with \mathbb{C} -constructible cohomology over the ring \mathbb{C}^h .

Theorem 3.12. *Let $\mathcal{M}, \mathcal{N} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_X^h)$. Then*

$$\text{SS}(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{M}, \mathcal{N})) = \text{SS}(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\text{gr}_h(\mathcal{M}), \text{gr}_h(\mathcal{N}))).$$

If moreover \mathcal{M} and \mathcal{N} are holonomic, then $\mathbf{R}\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{M}, \mathcal{N})$ is an object of $\mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^h)$.

Proof. Set $F = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{M}, \mathcal{N})$. Then F is cohomologically \hbar -complete by Theorem 1.9 and Proposition 1.5. Hence $\text{SS}(F) = \text{SS}(\text{gr}_h(F))$ by Proposition 1.15. Moreover, the finiteness of the stalks $\text{gr}_h(F_x)$ over \mathbb{C} implies the finiteness of F_x over \mathbb{C}^h by Theorem 1.11 applied with $X = \{\text{pt}\}$ and $\mathcal{A} = \mathbb{C}^h$. Q.E.D.

Applying Theorem 3.12, and [7, Theorem 11.3.3], we get:

Corollary 3.13. *Let $\mathcal{M} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_X^h)$. Then*

$$\text{SS}(\text{Sol}_h(\mathcal{M})) = \text{SS}(\text{DR}_h(\mathcal{M})) = \text{char}_h(\mathcal{M}).$$

If moreover \mathcal{M} is holonomic, then $\text{Sol}_h(\mathcal{M})$ and $\text{DR}_h(\mathcal{M})$ belong to $\mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^h)$.

Theorem 3.14. *Let $\mathcal{M} \in \mathbf{D}_{\text{hol}}^b(\mathcal{D}_X^h)$. Then there is a natural isomorphism in $\mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^h)$*

$$(3.8) \quad \text{Sol}_h(\mathcal{M}) \simeq \mathbf{D}'_h(\text{DR}_h(\mathcal{M})).$$

Proof. The natural \mathbb{C}^h -linear morphism

$$\begin{aligned} \text{R}\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{O}_X^h, \mathcal{M}) \otimes_{\mathbb{C}_X^h} \text{R}\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{M}, \mathcal{O}_X^h) \\ \rightarrow \text{R}\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{O}_X^h, \mathcal{O}_X^h) \simeq \mathbb{C}_X^h \end{aligned}$$

induces the morphism in $\mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^h)$

$$(3.9) \quad \alpha: \text{R}\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{M}, \mathcal{O}_X^h) \rightarrow \mathbf{D}'_h(\text{R}\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{O}_X^h, \mathcal{M})).$$

(Note that, choosing $\mathcal{M} = \mathcal{D}_X^h$, this morphism defines the morphism $\mathcal{O}_X^h \rightarrow \mathbf{D}'_h(\Omega_X^h[d_X])$.) The morphism (3.9) induces an isomorphism

$$\text{gr}_h(\alpha): \text{R}\mathcal{H}om_{\mathcal{D}_X^h}(\text{gr}_h(\mathcal{M}), \mathcal{O}_X) \rightarrow \mathbf{D}'(\text{R}\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{O}_X, \text{gr}_h(\mathcal{M}))).$$

It is thus an isomorphism by Corollary 1.17.

Q.E.D.

4 Formal extension of tempered functions

Let us start by reviewing after [9, Chapter 7] the construction of the sheaves of tempered distributions and of C^∞ -functions with temperate growth on the subanalytic site.

Let X be a real analytic manifold X . One says that a function $f \in \mathcal{C}_X^\infty(U)$ has *polynomial growth* at $p \in X$ if, for a local coordinate system (x_1, \dots, x_n) around p , there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$(4.1) \quad \sup_{x \in K \cap U} (\text{dist}(x, K \setminus U))^N |f(x)| < \infty.$$

One says that f is *tempered* at p if all its derivatives are of polynomial growth at p . One says that f is tempered if it is tempered at any point of X . One denotes by $\mathcal{C}_X^{\infty,t}(U)$ the \mathbb{C} -vector subspace of $\mathcal{C}^\infty(U)$ consisting of tempered functions. It then follows from a theorem of Lojaciewicz that $U \mapsto \mathcal{C}_X^{\infty,t}(U)$ ($U \in \text{Op}_{X_{\text{sa}}}$) is a sheaf on X_{sa} . We denote it by $\mathcal{C}_{X_{\text{sa}}}^{\infty,t}$ or simply $\mathcal{C}_X^{\infty,t}$ if there is no risk of confusion.

Lemma 4.1. *One has $H^j(U; \mathcal{C}_X^{\infty,t}) = 0$ for any $U \in \text{Op}_{X_{\text{sa}}}$ and any $j > 0$.*

This result is well-known (see [8, Chapter 1]), but we recall its proof for the reader's convenience.

Proof. Consider the full subcategory \mathcal{J} of $\text{Mod}(\mathbb{C}_{X_{\text{sa}}})$ consisting of sheaves F such that for any pair $U, V \in \text{Op}_{X_{\text{sa}}}$, the Mayer-Vietoris sequence

$$0 \rightarrow F(U \cup V) \rightarrow F(U) \oplus F(V) \rightarrow F(U \cap V) \rightarrow 0$$

is exact. Let us check that this category is injective with respect to the functor $\Gamma(U; \bullet)$. The only non obvious fact is that if $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ is an exact sequence and that F' belongs to \mathcal{J} , then $F(U) \rightarrow F''(U)$ is surjective. Let $t \in F''(U)$. There exist a finite covering $U = \bigcup_{i \in I} U_i$ and $s_i \in F(U_i)$ whose image in $F''(U_i)$ is $t|_{U_i}$. Then the proof goes by induction on the cardinal of I using the property of F' and standard arguments. To conclude, note that $\mathcal{C}_X^{\infty,t}$ belongs to \mathcal{J} thanks to Lojaciewicz's result (see [12]). Q.E.D.

Let $\mathcal{D}b_X$ be the sheaf of distributions on X . For $U \in \text{Op}_{X_{\text{sa}}}$, denote by $\mathcal{D}b_X^t(U)$ the space of tempered distributions on U , defined by the exact sequence

$$0 \rightarrow \Gamma_{X \setminus U}(X; \mathcal{D}b_X) \rightarrow \Gamma(X; \mathcal{D}b_X) \rightarrow \mathcal{D}b_X^t(U) \rightarrow 0.$$

Again, it follows from a theorem of Lojaciewicz that $U \mapsto \mathcal{D}b^t(U)$ is a sheaf on X_{sa} . We denote it by $\mathcal{D}b_{X_{\text{sa}}}^t$ or simply $\mathcal{D}b_X^t$ if there is no risk of confusion. The sheaf $\mathcal{D}b_X^t$ is quasi-injective, that is, the functor $\mathcal{H}om_{\mathbb{C}_{X_{\text{sa}}}}(\bullet, \mathcal{D}b_X^t)$ is exact in the category $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. Moreover, for $U \in \text{Op}_{X_{\text{sa}}}$, $\mathcal{H}om_{\mathbb{C}_{X_{\text{sa}}}}(\mathbb{C}_U, \mathcal{D}b_X^t)$ is also quasi-injective and $\text{R}\mathcal{H}om_{\mathbb{C}_{X_{\text{sa}}}}(\mathbb{C}_U, \mathcal{D}b_X^t)$ is concentrated in degree 0. Note that the sheaf

$$\Gamma_{[U]} \mathcal{D}b_X := \rho^{-1} \mathcal{H}om_{\mathbb{C}_{X_{\text{sa}}}}(\mathbb{C}_U, \mathcal{D}b_X^t)$$

is a \mathcal{C}_X^∞ -module, so that in particular $\text{R}\Gamma(V; \Gamma_{[U]} \mathcal{D}b_X)$ is concentrated in degree 0 for $V \subset X$ an open subset.

Formal extensions

By Proposition 2.5 the sheaves $\mathcal{C}_X^{\infty,t,\hbar}$, $\mathcal{D}b_X^{t,\hbar}$ and $\Gamma_{[U]}\mathcal{D}b_X$ are acyclic for the functor $(\bullet)^\hbar$. We set

$$\mathcal{C}_X^{\infty,t,\hbar} := (\mathcal{C}_X^{\infty,t})^\hbar, \quad \mathcal{D}b_X^{t,\hbar} := (\mathcal{D}b_X^t)^\hbar, \quad \Gamma_{[U]}\mathcal{D}b_X^\hbar := (\Gamma_{[U]}\mathcal{D}b_X)^\hbar.$$

Note that, by Lemmas 2.3 and 2.9,

$$\Gamma_{[U]}\mathcal{D}b_X^\hbar \simeq \rho^{-1}\mathcal{H}om_{\mathbb{C}_{X_{\text{sa}}}}(\mathbb{C}_U, \mathcal{D}b_X^{t,\hbar}).$$

By Proposition 2.2 we get:

Proposition 4.2. *The sheaves $\mathcal{C}_X^{\infty,t,\hbar}$, $\mathcal{D}b_X^{t,\hbar}$ and $\Gamma_{[U]}\mathcal{D}b_X^\hbar$ are cohomologically \hbar -complete.*

Now assume X is a complex manifold. Denote by \overline{X} the complex conjugate manifold and by $X^{\mathbb{R}}$ the underlying real analytic manifold, identified with the diagonal of $X \times \overline{X}$. One defines the sheaf (in fact, an object of the derived category) of tempered holomorphic functions by

$$\mathcal{O}_X^t := \mathbf{R}\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{C}_X^{\infty,t}) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{D}b_X^t).$$

Here and in the sequel, we write $\mathcal{C}_X^{\infty,t}$ and $\mathcal{D}b_X^t$ instead of $\mathcal{C}_{X^{\mathbb{R}}}^{\infty,t}$ and $\mathcal{D}b_{X^{\mathbb{R}}}^t$, respectively. We set

$$\mathcal{O}_X^{t,\hbar} := (\mathcal{O}_X^t)^{\text{R}\hbar},$$

a cohomologically \hbar -complete object of $\mathbf{D}^b(\mathbb{C}_{X_{\text{sa}}}^\hbar)$. By Lemma 2.3,

$$\mathcal{O}_X^{t,\hbar} \simeq \mathbf{R}\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{C}_X^{\infty,t,\hbar}) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{D}b_X^{t,\hbar}).$$

Note that $\text{gr}_\hbar(\mathcal{O}_X^{t,\hbar}) \simeq \mathcal{O}_X^t$ in $\mathbf{D}^b(\mathbb{C}_{X_{\text{sa}}})$.

5 Riemann-Hilbert correspondence

Let X be a complex analytic manifold. Consider the functors

$$\begin{aligned} \text{TH}(\bullet) : \mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X) &\rightarrow \mathbf{D}_{\text{rh}}^b(\mathcal{D}_X)^{\text{op}}, & F &\mapsto \rho^{-1}\mathbf{R}\mathcal{H}om_{\mathbb{C}_{X_{\text{sa}}}}(\rho_*F, \mathcal{O}_X^t), \\ \text{TH}_\hbar(\bullet) : \mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^\hbar) &\rightarrow \mathbf{D}^b(\mathcal{D}_X^\hbar)^{\text{op}}, & F &\mapsto \rho^{-1}\mathbf{R}\mathcal{H}om_{\mathbb{C}_{X_{\text{sa}}}^\hbar}(\rho_*F, \mathcal{O}_X^{t,\hbar}). \end{aligned}$$

The classical Riemann-Hilbert correspondence of Kashiwara [4] states that the functors Sol and TH are equivalences of categories between $\text{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X)$ and $\text{D}_{\text{rh}}^b(\mathcal{D}_X)^{\text{op}}$ quasi-inverse to each other. In order to obtain a similar statement for \mathbb{C}_X and \mathcal{D}_X replaced with \mathbb{C}_X^{\hbar} and \mathcal{D}_X^{\hbar} , respectively, we start by establishing some lemmas.

Lemma 5.1. *Let $\mathcal{M}, \mathcal{N} \in \text{D}_{\text{hol}}^b(\mathcal{D}_X^{\hbar})$. The natural morphism in $\text{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar})$*

$$\text{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \mathcal{N}) \rightarrow \text{R}\mathcal{H}om_{\mathbb{C}_X^{\hbar}}(\text{Sol}_{\hbar}(\mathcal{N}), \text{Sol}_{\hbar}(\mathcal{M}))$$

is an isomorphism.

Proof. Applying the functor gr_{\hbar} to this morphism, we get an isomorphism by the classical Riemann-Hilbert correspondence. Then the result follows from Corollary 1.17 and Theorem 3.12. Q.E.D.

Note that there is an isomorphism in $\text{D}^b(\mathcal{D}_X)$

$$(5.1) \quad \text{gr}_{\hbar}(\text{TH}_{\hbar}(F)) \simeq \text{TH}(\text{gr}_{\hbar}(F)).$$

Lemma 5.2. *The functor TH_{\hbar} induces a functor*

$$(5.2) \quad \text{TH}_{\hbar}: \text{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar}) \rightarrow \text{D}_{\text{rh}}^b(\mathcal{D}_X^{\hbar})^{\text{op}}.$$

Proof. Let $F \in \text{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar})$. By (5.1) and the classical Riemann-Hilbert correspondence we know that $\text{gr}_{\hbar}(\text{TH}_{\hbar}(F))$ is regular holonomic, and in particular coherent. It is thus left to prove that $\text{TH}_{\hbar}(F)$ is coherent. Note that our problem is of local nature.

We use the Dolbeault resolution of $\mathcal{O}_X^{t, \hbar}$ with coefficients in $\mathcal{D}_X^{t, \hbar}$ and we choose a resolution of F as given in Proposition A.1 (i). We find that $\text{TH}_{\hbar}(F)$ is isomorphic to a bounded complex \mathcal{M}^{\bullet} , where the \mathcal{M}^i are locally finite sums of sheaves of the type $\Gamma_{[U]} \mathcal{D}^{t, \hbar}$ with $U \in \text{Op}_{X_{\text{sa}}}$. It follows from Proposition 4.2 that $\text{TH}_{\hbar}(F)$ is cohomologically \hbar -complete, and we conclude by Theorem 1.11 with $\mathcal{A} = \mathcal{D}_X^{\hbar}$. Q.E.D.

Lemma 5.3. *We have $\text{R}\mathcal{H}om_{\rho! \mathcal{D}_X^{\hbar}}(\rho! \mathcal{O}_X^{\hbar}, \mathcal{O}_X^{t, \hbar}) \simeq \mathbb{C}_{X_{\text{sa}}}^{\hbar}$.*

Proof. This isomorphism is given by the sequence

$$\begin{aligned} \text{R}\mathcal{H}om_{\rho! \mathcal{D}_X^{\hbar}}(\rho! \mathcal{O}_X^{\hbar}, \mathcal{O}_X^{t, \hbar}) &\simeq \text{R}\mathcal{H}om_{\rho! \mathcal{D}_X}(\rho! \mathcal{O}_X, \mathcal{O}_X^{t, \hbar}) \\ &\simeq \text{R}\mathcal{H}om_{\rho! \mathcal{D}_X}(\rho! \mathcal{O}_X, \mathcal{O}_X^t)^{\text{Rh}} \\ &\simeq (\rho_* \text{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X))^{\text{Rh}} \simeq (\mathbb{C}_{X_{\text{sa}}})^{\text{Rh}} \simeq \mathbb{C}_{X_{\text{sa}}}^{\hbar}, \end{aligned}$$

where the first isomorphism is an extension of scalars, the second one is Lemma 2.3 and the third one is given by the adjunction between $\rho_!$ and ρ^{-1} .
Q.E.D.

Theorem 5.4. *The functors Sol_h and TH_h are equivalences of categories between $\text{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^h)$ and $\text{D}_{\text{rh}}^b(\mathcal{D}_X^h)^{\text{op}}$ quasi-inverse to each other.*

Proof. In view of Lemma 5.1, we know that the functor Sol_h is fully faithful. It is then enough to show that $\text{Sol}_h(\text{TH}_h(F)) \simeq F$ for $F \in \text{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^h)$. Since we already know by Lemma 5.2 that $\text{TH}_h(F)$ is holonomic, we may use (3.8). We have the sequence of isomorphisms:

$$\begin{aligned} \rho_* \text{R}\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{O}_X^h, \text{TH}_h(F)) &= \rho_* \text{R}\mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{O}_X^h, \rho^{-1} \text{R}\mathcal{H}om_{\mathbb{C}_{X\text{sa}}^h}(\rho_* F, \mathcal{O}_X^{t,h})) \\ &\simeq \text{R}\mathcal{H}om_{\rho_! \mathcal{D}_X^h}(\rho_! \mathcal{O}_X^h, \text{R}\mathcal{H}om_{\mathbb{C}_{X\text{sa}}^h}(\rho_* F, \mathcal{O}_X^{t,h})) \\ &\simeq \text{R}\mathcal{H}om_{\mathbb{C}_{X\text{sa}}^h}(\rho_* F, \text{R}\mathcal{H}om_{\rho_! \mathcal{D}_X^h}(\rho_! \mathcal{O}_X^h, \mathcal{O}_X^{t,h})) \\ &\simeq \text{R}\mathcal{H}om_{\mathbb{C}_{X\text{sa}}^h}(\rho_* F, \mathbb{C}_{X\text{sa}}^h) \simeq \text{R}\mathcal{H}om_{\mathbb{C}_{X\text{sa}}^h}(\rho_* F, \rho_* \mathbb{C}_X^h) \\ &\simeq \rho_* D_h' F, \end{aligned}$$

where we have used the adjunction between $\rho_!$ and ρ^{-1} , the isomorphism of Lemma 5.3 and the commutation of ρ_* with $\text{R}\mathcal{H}om$. One concludes by recalling the isomorphism of functors $\rho^{-1} \rho_* \simeq \text{id}$.
Q.E.D.

t -structure

Recall the definition of the middle perversity t -structure for complex constructible sheaves. Let \mathbb{K} denote either the field \mathbb{C} or the ring \mathbb{C}^h . For $F \in \text{D}_{\mathbb{C}\text{-c}}^b(\mathbb{K}_X)$, we have $F \in {}^p\text{D}_{\mathbb{C}\text{-c}}^{\leq 0}(\mathbb{K}_X)$ if and only if

$$(5.3) \quad \forall i \in \mathbb{Z} \quad \dim \text{supp } H^i(F) \leq d_X - i,$$

and $F \in {}^p\text{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{K}_X)$ if and only if, for any locally closed complex analytic subset $S \subset X$,

$$(5.4) \quad H_S^i(F) = 0 \text{ for all } i < d_X - \dim(S).$$

With the above convention, the de Rham functor

$$\text{DR}: \text{D}_{\text{hol}}^b(\mathcal{D}_X) \rightarrow {}^p\text{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X)$$

is t -exact.

Theorem 5.5. *The de Rham functor $\mathrm{DR}_{\hbar}: \mathrm{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_X^{\hbar}) \rightarrow {}^p\mathrm{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(\mathbb{C}_X^{\hbar})$ is t -exact.*

Proof. (i) Let $\mathcal{M} \in \mathrm{D}_{\mathrm{hol}}^{\leq 0}(\mathcal{D}_X^{\hbar})$. Let us prove that $\mathrm{DR}_{\hbar}\mathcal{M} \in {}^p\mathrm{D}_{\mathbb{C}\text{-c}}^{\leq 0}(\mathbb{C}_X^{\hbar})$. Since $\mathrm{DR}_{\hbar}\mathcal{M}$ is constructible, Proposition 1.19 shows that it is enough to check (5.3) for $\mathrm{gr}_{\hbar}(\mathrm{DR}_{\hbar}\mathcal{M}) \simeq \mathrm{DR}(\mathrm{gr}_{\hbar}\mathcal{M})$. In other words, it is enough to check that $\mathrm{DR}(\mathrm{gr}_{\hbar}\mathcal{M}) \in {}^p\mathrm{D}_{\mathbb{C}\text{-c}}^{\leq 0}(\mathbb{C}_X)$. Since $\mathrm{gr}_{\hbar}\mathcal{M} \in \mathrm{D}_{\mathrm{hol}}^{\leq 0}(\mathcal{D}_X)$, this result follows from the t -exactness of the functor DR .

(ii) Let $\mathcal{M} \in \mathrm{D}_{\mathrm{hol}}^{\geq 0}(\mathcal{D}_X^{\hbar})$. Let us prove that $\mathrm{DR}_{\hbar}\mathcal{M} \in {}^p\mathrm{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{C}_X^{\hbar})$. We set $\mathcal{N} = (H^0\mathcal{M})_{\hbar\text{-tor}}$. We have a morphism $u: \mathcal{N} \rightarrow \mathcal{M}$ induced by $H^0\mathcal{M} \rightarrow \mathcal{M}$ and we let \mathcal{M}' be the mapping cone of u . We have a distinguished triangle

$$\mathrm{DR}_{\hbar}\mathcal{N} \rightarrow \mathrm{DR}_{\hbar}\mathcal{M} \rightarrow \mathrm{DR}_{\hbar}\mathcal{M}' \xrightarrow{+1}$$

so that it is enough to show that $\mathrm{DR}_{\hbar}\mathcal{N}$ and $\mathrm{DR}_{\hbar}\mathcal{M}'$ belong to ${}^p\mathrm{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{C}_X^{\hbar})$.

(a) By Proposition 3.6 (ii) and Proposition 3.8, \mathcal{N} is holonomic as a \mathcal{D}_X -module. Hence $\mathrm{DR}_{\hbar}\mathcal{N} \simeq \mathrm{DR}\mathcal{N}$ is a perverse sheaf (over \mathbb{C}) and satisfies (5.4). Since (5.4) does not depend on the coefficient ring, $\mathrm{DR}_{\hbar}\mathcal{N} \in {}^p\mathrm{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{C}_X^{\hbar})$.

(b) We note that $H^0\mathcal{M}' \simeq (H^0\mathcal{M})_{\hbar\text{-tf}}$. Hence by Proposition 1.14, $\mathrm{gr}_{\hbar}\mathcal{M}' \in \mathrm{D}_{\mathrm{hol}}^{\geq 0}(\mathcal{D}_X)$ and $\mathrm{DR}(\mathrm{gr}_{\hbar}\mathcal{M}') \in {}^p\mathrm{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{C}_X)$, that is, $\mathrm{DR}(\mathrm{gr}_{\hbar}\mathcal{M}')$ satisfies (5.4). Let $S \subset X$ be a locally closed complex subanalytic subset. We have

$$\mathrm{R}\Gamma_S(\mathrm{DR}(\mathrm{gr}_{\hbar}\mathcal{M}')) \simeq \mathrm{gr}_{\hbar}(\mathrm{R}\Gamma_S(\mathrm{DR}_{\hbar}\mathcal{M}'))$$

and it follows from Proposition 1.19 that $\mathrm{DR}_{\hbar}\mathcal{M}'$ also satisfies (5.4) and thus belongs to ${}^p\mathrm{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{C}_X^{\hbar})$. Q.E.D.

6 Duality and \hbar -torsion

The duality functors \mathbb{D} on $\mathrm{D}_{\mathrm{rh}}(\mathcal{D}_X)$ and D' on ${}^p\mathrm{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(\mathbb{C}_X)$ are t -exact. We will discuss here the finer t -structures needed in order to obtain a similar result when replacing \mathbb{C}_X and \mathcal{D}_X by their formal extensions \mathbb{C}_X^{\hbar} and \mathcal{D}_X^{\hbar} .

Following [1, Chapter I.2], let us start by recalling some facts related to torsion pairs and t -structures. We need in particular Proposition 6.2 below, which can also be found in [2].

Definition 6.1. Let \mathcal{C} be an abelian category. A torsion pair on \mathcal{C} is a pair $(\mathcal{C}_{\text{tor}}, \mathcal{C}_{\text{tf}})$ of full subcategories such that

- (i) for all objects T in \mathcal{C}_{tor} and F in \mathcal{C}_{tf} , we have $\text{Hom}_{\mathcal{C}}(T, F) = 0$,
- (ii) for any object M in \mathcal{C} , there are objects M_{tor} in \mathcal{C}_{tor} and M_{tf} in \mathcal{C}_{tf} and a short exact sequence $0 \rightarrow M_{\text{tor}} \rightarrow M \rightarrow M_{\text{tf}} \rightarrow 0$.

Proposition 6.2. Let \mathbf{D} be a triangulated category endowed with a t -structure $({}^p\mathbf{D}^{\leq 0}, {}^p\mathbf{D}^{\geq 0})$. Let us denote its heart by \mathcal{C} and its cohomology functors by ${}^pH^i: \mathbf{D} \rightarrow \mathcal{C}$. Suppose that \mathcal{C} is endowed with a torsion pair $(\mathcal{C}_{\text{tor}}, \mathcal{C}_{\text{tf}})$. Then we can define a new t -structure $({}^\pi\mathbf{D}^{\leq 0}, {}^\pi\mathbf{D}^{\geq 0})$ on \mathbf{D} by setting:

$$\begin{aligned} {}^\pi\mathbf{D}^{\leq 0} &= \{M \in {}^p\mathbf{D}^{\leq 1} : {}^pH^1(M) \in \mathcal{C}_{\text{tor}}\}, \\ {}^\pi\mathbf{D}^{\geq 0} &= \{M \in {}^p\mathbf{D}^{\geq 0} : {}^pH^0(M) \in \mathcal{C}_{\text{tf}}\}. \end{aligned}$$

With the notations of Definition 3.2, there is a natural torsion pair attached to $\text{Mod}(\mathcal{D}_X^{\hbar})$ given by the full subcategories

$$\begin{aligned} \text{Mod}(\mathcal{D}_X^{\hbar})_{\hbar\text{-tor}} &= \{\mathcal{M} : \mathcal{M}_{\hbar\text{-tor}} \xrightarrow{\sim} \mathcal{M}\}, \\ \text{Mod}(\mathcal{D}_X^{\hbar})_{\hbar\text{-tf}} &= \{\mathcal{M} : \mathcal{M} \xrightarrow{\sim} \mathcal{M}_{\hbar\text{-tf}}\}. \end{aligned}$$

Definition 6.3. (a) We call the torsion pair on $\text{Mod}(\mathcal{D}_X^{\hbar})$ defined above, the \hbar -torsion pair.

- (b) We denote by $(\mathbf{D}^{\leq 0}(\mathcal{D}_X^{\hbar}), \mathbf{D}^{\geq 0}(\mathcal{D}_X^{\hbar}))$ the natural t -structure on $\mathbf{D}(\mathcal{D}_X^{\hbar})$.
- (c) We denote by $({}^t\mathbf{D}^{\leq 0}(\mathcal{D}_X^{\hbar}), {}^t\mathbf{D}^{\geq 0}(\mathcal{D}_X^{\hbar}))$ the t -structure on $\mathbf{D}^b(\mathcal{D}_X^{\hbar})$ associated via Proposition 6.2 with the \hbar -torsion pair on $\text{Mod}(\mathcal{D}_X^{\hbar})$.

Proposition 1.14 implies the following equivalences for $\mathcal{M} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_X^{\hbar})$:

$$(6.1) \quad \mathcal{M} \in {}^t\mathbf{D}^{\geq 0}(\mathcal{D}_X^{\hbar}) \iff \text{gr}_{\hbar} \mathcal{M} \in \mathbf{D}^{\geq 0}(\mathcal{D}_X),$$

$$(6.2) \quad \mathcal{M} \in \mathbf{D}^{\leq 0}(\mathcal{D}_X^{\hbar}) \iff \text{gr}_{\hbar} \mathcal{M} \in \mathbf{D}^{\leq 0}(\mathcal{D}_X).$$

Proposition 6.4. Let \mathcal{M} be a holonomic \mathcal{D}_X^{\hbar} -module.

- (i) If \mathcal{M} has no \hbar -torsion, then $\mathbb{D}_{\hbar} \mathcal{M}$ is concentrated in degree 0 and has no \hbar -torsion.
- (ii) If \mathcal{M} is an \hbar -torsion module, then $\mathbb{D}_{\hbar} \mathcal{M}$ is concentrated in degree 1 and is an \hbar -torsion module.

Proof. By (1.2) we have $\mathrm{gr}_{\hbar}(\mathbb{D}_{\hbar}\mathcal{M}) \simeq \mathbb{D}(\mathrm{gr}_{\hbar}\mathcal{M})$. Since $\mathrm{gr}_{\hbar}\mathcal{M}$ is concentrated in degrees 0 and -1 , with holonomic cohomology, $\mathbb{D}(\mathrm{gr}_{\hbar}\mathcal{M})$ is concentrated in degrees 0 and 1. By Proposition 1.14, $\mathbb{D}_{\hbar}\mathcal{M}$ itself is concentrated in degrees 0 and 1 and $H^0(\mathbb{D}_{\hbar}\mathcal{M})$ has no \hbar -torsion.

(i) The short exact sequence

$$0 \rightarrow \mathcal{M} \xrightarrow{\hbar} \mathcal{M} \rightarrow \mathcal{M}/\hbar\mathcal{M} \rightarrow 0$$

induces the long exact sequence

$$\cdots \rightarrow H^1(\mathbb{D}_{\hbar}(\mathcal{M}/\hbar\mathcal{M})) \rightarrow H^1(\mathbb{D}_{\hbar}\mathcal{M}) \xrightarrow{\hbar} H^1(\mathbb{D}_{\hbar}\mathcal{M}) \rightarrow 0.$$

By Nakayama's lemma $H^1(\mathbb{D}_{\hbar}\mathcal{M}) = 0$ as required.

(ii) Since \mathcal{M} is locally annihilated by some power of \hbar , the cohomology groups $H^i(\mathbb{D}_{\hbar}\mathcal{M})$ also are \hbar -torsion modules. As $H^0(\mathbb{D}_{\hbar}\mathcal{M})$ has no \hbar -torsion, we get $H^0(\mathbb{D}_{\hbar}\mathcal{M}) = 0$. Q.E.D.

Theorem 6.5. *The duality functor $\mathbb{D}_{\hbar}: \mathbf{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_X^{\hbar})^{\mathrm{op}} \rightarrow {}^t\mathbf{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_X^{\hbar})$ is t -exact. In other words, \mathbb{D}_{\hbar} interchanges $\mathbf{D}_{\mathrm{hol}}^{\leq 0}(\mathcal{D}_X^{\hbar})$ with ${}^t\mathbf{D}_{\mathrm{hol}}^{\geq 0}(\mathcal{D}_X^{\hbar})$ and $\mathbf{D}_{\mathrm{hol}}^{\geq 0}(\mathcal{D}_X^{\hbar})$ with ${}^t\mathbf{D}_{\mathrm{hol}}^{\leq 0}(\mathcal{D}_X^{\hbar})$.*

Proof. (i) Let us first prove for $\mathcal{M} \in \mathbf{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_X^{\hbar})$:

$$(6.3) \quad \mathcal{M} \in \mathbf{D}_{\mathrm{hol}}^{\leq 0}(\mathcal{D}_X^{\hbar}) \iff \mathbb{D}_{\hbar}(\mathcal{M}) \in {}^t\mathbf{D}_{\mathrm{hol}}^{\geq 0}(\mathcal{D}_X^{\hbar}).$$

By (1.2) we have $\mathrm{gr}_{\hbar}(\mathbb{D}_{\hbar}\mathcal{M}) \simeq \mathbb{D}(\mathrm{gr}_{\hbar}\mathcal{M})$ and we know that the analog of (6.3) holds true for \mathcal{D}_X -modules:

$$\mathcal{N} \in \mathbf{D}_{\mathrm{hol}}^{\leq 0}(\mathcal{D}_X) \iff \mathbb{D}(\mathcal{N}) \in \mathbf{D}_{\mathrm{hol}}^{\geq 0}(\mathcal{D}_X).$$

Hence (6.3) follows easily from (6.1) and (6.2).

(ii) We recall the general fact for a t -structure $(\mathbf{D}, \mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ and $A \in \mathbf{D}$:

$$\begin{aligned} A \in \mathbf{D}^{\leq 0} &\iff \forall B \in \mathbf{D}^{\geq 1} \mathrm{Hom}(A, B) = 0, \\ A \in \mathbf{D}^{\geq 0} &\iff \forall B \in \mathbf{D}^{\leq -1} \mathrm{Hom}(B, A) = 0. \end{aligned}$$

Since \mathbb{D}_{\hbar} is an involutive equivalence of categories we deduce from (6.3) the dual statement:

$$\mathcal{M} \in \mathbf{D}_{\mathrm{hol}}^{\geq 0}(\mathcal{D}_X^{\hbar}) \iff \mathbb{D}_{\hbar}(\mathcal{M}) \in {}^t\mathbf{D}_{\mathrm{hol}}^{\leq 0}(\mathcal{D}_X^{\hbar}).$$

Q.E.D.

Remark 6.6. The above result can be stated as follows in the language of quasi-abelian categories of [16]. We will follow the same notations as in [6, Chapter 2]. The category $\mathcal{C} = \text{Mod}(\mathcal{D}_X^h)_{h\text{-tf}}$ is quasi-abelian. Hence its derived category has a natural generalized t -structure $(\mathbf{D}^{\leq s}(\mathcal{C}), \mathbf{D}^{> s-1}(\mathcal{C}))_{s \in \frac{1}{2}\mathbb{Z}}$. Note that $\mathbf{D}^{[-1/2, 0]}(\mathcal{C})$ is equivalent to $\text{Mod}(\mathcal{D}_X^h)$, and that $\mathbf{D}^{[0, 1/2]}(\mathcal{C})$ is equivalent to the heart of ${}^t\mathbf{D}^b(\mathcal{D}_X^h)$. Then Theorem 6.5 states that the duality functor \mathbb{D}_h is t -exact on $\mathbf{D}_{\text{hol}}^b(\mathcal{C})$.

Consider the full subcategories of $\text{Perv}(\mathbb{C}_X^h)$

$$\text{Perv}(\mathbb{C}_X^h)_{h\text{-tor}} = \{F : \text{locally } \hbar^N F = 0 \text{ for some } N \in \mathbb{N}\},$$

$$\text{Perv}(\mathbb{C}_X^h)_{h\text{-tf}} = \{F : F \text{ has no non zero subobjects in } \text{Perv}(\mathbb{C}_X^h)_{h\text{-tor}}\}.$$

Lemma 6.7. (i) *Let $F \in \text{Perv}(\mathbb{C}_X^h)$. Then the inductive system of sub-perverse sheaves $\text{Ker}(\hbar^n : F \rightarrow F)$ is locally stationary.*

(ii) *The pair $(\text{Perv}(\mathbb{C}_X^h)_{h\text{-tor}}, \text{Perv}(\mathbb{C}_X^h)_{h\text{-tf}})$ is a torsion pair.*

Proof. (i) Set $\mathcal{M} = \mathbb{D}_h \text{TH}_h(F)$. By the Riemann-Hilbert correspondence, one has $\text{Ker}(\hbar^n : F \rightarrow F) \simeq \text{DR}_h(\text{Ker}(\hbar^n : \mathcal{M} \rightarrow \mathcal{M}))$. Since \mathcal{M} is coherent, the inductive system $\text{Ker}(\hbar^n : \mathcal{M} \rightarrow \mathcal{M})$ is locally stationary. Hence so is the system $\text{Ker}(\hbar^n : F \rightarrow F)$.

(ii) By (i) it makes to define for $F \in \text{Perv}(\mathbb{C}_X^h)$:

$$F_{h\text{-tor}} = \bigcup_n \text{Ker}(\hbar^n : F \rightarrow F), \quad F_{h\text{-tf}} = F / F_{h\text{-tor}}.$$

It is easy to check that $F_{h\text{-tor}} \in \text{Perv}(\mathbb{C}_X^h)_{h\text{-tor}}$ and $F_{h\text{-tf}} \in \text{Perv}(\mathbb{C}_X^h)_{h\text{-tf}}$. Then property (ii) in Definition 6.1 is clear. For property (i) let $u : F \rightarrow G$ be a morphism in $\text{Perv}(\mathbb{C}_X^h)$ with $F \in \text{Perv}(\mathbb{C}_X^h)_{h\text{-tor}}$ and $G \in \text{Perv}(\mathbb{C}_X^h)_{h\text{-tf}}$. Then $\text{Im } u$ also is in $\text{Perv}(\mathbb{C}_X^h)_{h\text{-tor}}$ and so it is zero by definition of $\text{Perv}(\mathbb{C}_X^h)_{h\text{-tf}}$. Q.E.D.

Denote by $(\pi \mathbf{D}_{\mathbb{C}\text{-c}}^{\leq 0}(\mathbb{C}_X^h), \pi \mathbf{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{C}_X^h))$ the t -structure on $\mathbf{D}_{\mathbb{C}\text{-c}}(\mathbb{C}_X^h)$ induced by the perversity t -structure and this torsion pair as in Proposition 6.2. We also set $\pi \text{Perv}(\mathbb{C}_X^h) = \pi \mathbf{D}_{\mathbb{C}\text{-c}}^{\leq 0}(\mathbb{C}_X^h) \cap \pi \mathbf{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{C}_X^h)$.

Corollary 6.8. *There is a quasi-commutative diagram of t -exact functors*

$$\begin{array}{ccc} \mathbf{D}_{\text{hol}}^b(\mathcal{D}_X^h)^{\text{op}} & \xrightarrow{\text{DR}_h} & {}^p\mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^h)^{\text{op}} \\ \downarrow \mathbb{D}_h & & \downarrow \mathbf{D}'_h \\ {}^t\mathbf{D}_{\text{hol}}^b(\mathcal{D}_X^h) & \xrightarrow{\text{DR}_h} & \pi \mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^h) \end{array}$$

where the duality functors are equivalences of categories and the de Rham functors become equivalences when restricted to the subcategories of regular objects.

Example 6.9. Let $X = \mathbb{C}$, $U = X \setminus \{0\}$ and denote by $j: U \hookrightarrow X$ the embedding. Let L be the local system on U with stalk \mathbb{C}^h and monodromy $1 + h$. The sheaf $Rj_*L \simeq D'_h(j_!(D'_hL))$ is perverse for both t -structures, as is the sheaf $H^0(Rj_*L) = j_*L \simeq j_!L$. The sheaf $H^1(Rj_*L) \simeq \mathbb{C}_{\{0\}}$ has h -torsion. From the distinguished triangle $j_*L \rightarrow Rj_*L \rightarrow \mathbb{C}_{\{0\}}[-1] \xrightarrow{+1}$, one gets the short exact sequences

$$\begin{aligned} 0 \rightarrow j_*L \rightarrow Rj_*L \rightarrow \mathbb{C}_{\{0\}}[-1] \rightarrow 0 & \text{ in } \text{Perv}(\mathbb{C}_X^h), \\ 0 \rightarrow \mathbb{C}_{\{0\}}[-2] \rightarrow j_*L \rightarrow Rj_*L \rightarrow 0 & \text{ in } {}^\pi\text{Perv}(\mathbb{C}_X^h). \end{aligned}$$

7 $\mathcal{D}((h))$ -modules

Denote by

$$\mathbb{C}^{h,\text{loc}} := \mathbb{C}((h)) = \mathbb{C}[h^{-1}, h]$$

the field of Laurent series in h , that is the fraction field of \mathbb{C}^h . Recall the exact functor

$$(7.1) \quad (\cdot)^{\text{loc}}: \text{Mod}(\mathbb{C}_X^h) \rightarrow \text{Mod}(\mathbb{C}_X^{h,\text{loc}}), \quad F \mapsto \mathbb{C}_X^{h,\text{loc}} \otimes_{\mathbb{C}_X^h} F,$$

and note that by [7, Proposition 5.4.14] one has the estimate

$$(7.2) \quad \text{SS}(F^{\text{loc}}) \subset \text{SS}(F).$$

For $G \in \text{D}^b(\mathbb{C}_X)$, we write $G^{h,\text{loc}}$ instead of $(G^h)^{\text{loc}}$. We will consider in particular

$$\mathcal{O}_X^{h,\text{loc}} = \mathcal{O}_X((h)), \quad \mathcal{D}_X^{h,\text{loc}} = \mathcal{D}_X((h)).$$

Lemma 7.1. *Let \mathcal{M} be a coherent $\mathcal{D}_X^{h,\text{loc}}$ -module. Then \mathcal{M} is pseudo-coherent over \mathcal{D}_X^h . In other word, if $\mathcal{L} \subset \mathcal{M}$ is a finitely generated \mathcal{D}_X^h -module, then \mathcal{L} is \mathcal{D}_X^h -coherent.*

Proof. The proof follows from [5, Appendix. A1].

Q.E.D.

Definition 7.2. A lattice \mathcal{L} of a coherent $\mathcal{D}_X^{h,\text{loc}}$ -module \mathcal{M} is a coherent \mathcal{D}_X^h -submodule of \mathcal{M} which generates it.

Since \mathcal{M} has no \hbar -torsion, any of its lattices has no \hbar -torsion. In particular, one has $\mathcal{M} \simeq \mathcal{L}^{\text{loc}}$ and $\text{gr}_{\hbar} \mathcal{L} \simeq \mathcal{L}_0 = \mathcal{L}/\hbar \mathcal{L}$.

It follows from Lemma 7.1 that lattices locally exist: for a local system of generators (m_1, \dots, m_N) of \mathcal{M} , define \mathcal{L} as the \mathcal{D}_X^{\hbar} -submodule with the same generators.

Lemma 7.3. *Let $0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0$ be an exact sequence of coherent $\mathcal{D}_X^{\hbar, \text{loc}}$ -modules. Locally there exist lattices $\mathcal{L}', \mathcal{L}, \mathcal{L}''$ of $\mathcal{M}', \mathcal{M}, \mathcal{M}''$, respectively, inducing an exact sequence of \mathcal{D}_X^{\hbar} -modules*

$$0 \rightarrow \mathcal{L}' \rightarrow \mathcal{L} \rightarrow \mathcal{L}'' \rightarrow 0.$$

Proof. Let \mathcal{L} be a lattice of \mathcal{M} and let \mathcal{L}'' be its image in \mathcal{M}'' . We set $\mathcal{L}' := \mathcal{L} \cap \mathcal{M}'$. These sub- \mathcal{D}_X^{\hbar} -modules give rise to an exact sequence.

Since \mathcal{L}'' is of finite type over \mathcal{D}_X^{\hbar} , it is a lattice of \mathcal{M}'' . Let us show that \mathcal{L}' is a lattice of \mathcal{M}' . Being the kernel of a morphism $\mathcal{L} \rightarrow \mathcal{L}''$ between coherent \mathcal{D}_X^{\hbar} -modules, \mathcal{L}' is coherent. To show that \mathcal{L}' generates \mathcal{M}' , note that any $m' \in \mathcal{M}' \subset \mathcal{M}$ may be written as $m' = \hbar^{-N} m$ for some $N \geq 0$ and $m \in \mathcal{L}$. Hence $m = \hbar^N m' \in \mathcal{M}' \cap \mathcal{L} = \mathcal{L}'$. Q.E.D.

For an abelian category \mathcal{C} , we denote by $K(\mathcal{C})$ its Grothendieck group. For an object M of \mathcal{C} , we denote by $[M]$ its class in $K(\mathcal{C})$. We let $\mathcal{K}(\mathcal{D}_X)$ be the sheaf on X associated to the presheaf

$$U \mapsto K(\text{Mod}_{\text{coh}}(\mathcal{D}_X|_U)).$$

We define $\mathcal{K}(\mathcal{D}_X^{\hbar, \text{loc}})$ in the same way.

Lemma 7.4. *Let \mathcal{L} be a coherent \mathcal{D}_X^{\hbar} -module without \hbar -torsion. Then, for any $i > 0$, the \mathcal{D}_X -module $\mathcal{L}/\hbar^i \mathcal{L}$ is coherent, and we have the equality $[\mathcal{L}/\hbar^i \mathcal{L}] = i \cdot [\text{gr}_{\hbar}(\mathcal{L})]$ in $K(\text{Mod}_{\text{coh}}(\mathcal{D}_X))$.*

Proof. Since the functor $(\cdot) \otimes_{\mathbb{C}^{\hbar}} \mathbb{C}^{\hbar}/\hbar^i \mathbb{C}^{\hbar}$ is right exact, $\mathcal{L}/\hbar^i \mathcal{L}$ is a coherent \mathcal{D}_X -module. Since \mathcal{L} has no \hbar -torsion, multiplication by \hbar^i induces an isomorphism $\mathcal{L}/\hbar \mathcal{L} \xrightarrow{\sim} \hbar^i \mathcal{L}/\hbar^{i+1} \mathcal{L}$. We conclude by induction on i with the exact sequence

$$0 \rightarrow \hbar^i \mathcal{L}/\hbar^{i+1} \mathcal{L} \rightarrow \mathcal{L}/\hbar^{i+1} \mathcal{L} \rightarrow \mathcal{L}/\hbar^i \mathcal{L} \rightarrow 0.$$

Q.E.D.

Lemma 7.5. *For $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar, \text{loc}})$, $U \subset X$ an open set and $\mathcal{L} \subset \mathcal{M}|_U$ a lattice of $\mathcal{M}|_U$, the class $[\text{gr}_{\hbar}(\mathcal{L})] \in \text{K}(\text{Mod}_{\text{coh}}(\mathcal{D}_X|_U))$ only depends on \mathcal{M} . This defines a morphism of abelian sheaves $\mathcal{K}(\mathcal{D}_X^{\hbar, \text{loc}}) \rightarrow \mathcal{K}(\mathcal{D}_X)$.*

Proof. (i) We first prove that $[\text{gr}_{\hbar}(\mathcal{L})]$ only depends on \mathcal{M} . We consider another lattice \mathcal{L}' of $\mathcal{M}|_U$. Since \mathcal{L} is a \mathcal{D}_X^{\hbar} -module of finite type, and \mathcal{L}' generates \mathcal{M} , there exists $n > 1$ such that $\mathcal{L} \subset \hbar^{-n} \mathcal{L}'$. Similarly, there exists $m > 1$ with $\mathcal{L}' \subset \hbar^{-m} \mathcal{L}$, so that we have the inclusions

$$\hbar^{m+n+2} \mathcal{L} \subset \hbar^{m+n+1} \mathcal{L} \subset \hbar^{m+1} \mathcal{L}' \subset \hbar^m \mathcal{L}' \subset \mathcal{L}.$$

Any inclusion $A \subset B \subset C$ yields an identity $[C/A] = [C/B] + [B/A]$ in the Grothendieck group, and we obtain in particular:

$$\begin{aligned} [\hbar^m \mathcal{L}' / \hbar^{m+n+1} \mathcal{L}] &= [\hbar^m \mathcal{L}' / \hbar^{m+1} \mathcal{L}'] + [\hbar^{m+1} \mathcal{L}' / \hbar^{m+n+1} \mathcal{L}] \\ [\mathcal{L} / \hbar^{m+n+1} \mathcal{L}] &= [\mathcal{L} / \hbar^{m+1} \mathcal{L}'] + [\hbar^{m+1} \mathcal{L}' / \hbar^{m+n+1} \mathcal{L}] \\ [\mathcal{L} / \hbar^{m+n+2} \mathcal{L}] &= [\mathcal{L} / \hbar^{m+1} \mathcal{L}'] + [\hbar^{m+1} \mathcal{L}' / \hbar^{m+n+2} \mathcal{L}]. \end{aligned}$$

Since our modules have no \hbar -torsion, we have isomorphisms of the type $\hbar^k \mathcal{M}_1 / \hbar^k \mathcal{M}_2 \simeq \mathcal{M}_1 / \mathcal{M}_2$. Then Lemma 7.4 and the above equalities give:

$$\begin{aligned} [\mathcal{L}' / \hbar^{n+1} \mathcal{L}] &= [\text{gr}_{\hbar}(\mathcal{L}')] + [\mathcal{L}' / \hbar^n \mathcal{L}] \\ (m+n+1)[\text{gr}_{\hbar}(\mathcal{L})] &= [\mathcal{L} / \hbar^{m+1} \mathcal{L}'] + [\mathcal{L}' / \hbar^n \mathcal{L}] \\ (m+n+2)[\text{gr}_{\hbar}(\mathcal{L})] &= [\mathcal{L} / \hbar^{m+1} \mathcal{L}'] + [\mathcal{L}' / \hbar^{n+1} \mathcal{L}]. \end{aligned}$$

A suitable combination of these lines gives $[\text{gr}_{\hbar}(\mathcal{L})] = [\text{gr}_{\hbar}(\mathcal{L}')]$, as desired.

(ii) Now we consider an open subset $V \subset X$ and $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar, \text{loc}}|_V)$. We choose an open covering $\{U_i\}_{i \in I}$ of V such that for each $i \in I$ $\mathcal{M}|_{U_i}$ admits a lattice, say \mathcal{L}^i . We have seen that $[\text{gr}_{\hbar}(\mathcal{L}^i)] \in \text{K}(\text{Mod}_{\text{coh}}(\mathcal{D}_X|_{U_i}))$ only depends on \mathcal{M} . This implies that

$$[\text{gr}_{\hbar}(\mathcal{L}^i)]|_{U_{i,j}} = [\text{gr}_{\hbar}(\mathcal{L}^j)]|_{U_{i,j}} \text{ in } \text{K}(\text{Mod}_{\text{coh}}(\mathcal{D}_X|_{U_{i,j}})).$$

Hence the $[\text{gr}_{\hbar}(\mathcal{L}^i)]$'s define a section, say $c(\mathcal{M})$, of $\mathcal{K}(\mathcal{D}_X)$ over V . By Lemma 7.3, $c(\mathcal{M})$ only depends on the class $[\mathcal{M}]$ in $\text{K}(\text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar, \text{loc}}|_V))$, and $\mathcal{M} \mapsto c(\mathcal{M})$ induces the morphism $\mathcal{K}(\mathcal{D}_X^{\hbar, \text{loc}}) \rightarrow \mathcal{K}(\mathcal{D}_X)$. Q.E.D.

By Lemma 7.5, the following definition is well posed.

Definition 7.6. Let \mathcal{M} be a coherent $\mathcal{D}_X^{\hbar, \text{loc}}$ -module. For $\mathcal{L} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar})$ a (local) lattice, the characteristic variety of \mathcal{M} is defined by

$$\text{char}_{\hbar, \text{loc}}(\mathcal{M}) = \text{char}_{\hbar}(\mathcal{L}).$$

For $\mathcal{M} \in \text{D}_{\text{coh}}^{\text{b}}(\mathcal{D}_X^{\hbar, \text{loc}})$, one sets $\text{char}_{\hbar, \text{loc}}(\mathcal{M}) = \bigcup_j \text{char}_{\hbar, \text{loc}}(H^j(\mathcal{M}))$.

Proposition 7.7. *The characteristic variety $\text{char}_{\hbar, \text{loc}}(\cdot)$ is additive both on $\text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar, \text{loc}})$ and on $\text{D}^{\text{b}}(\mathcal{D}_X^{\hbar, \text{loc}})$.*

Proof. This follows from Proposition 3.6 (ii) and Lemma 7.3. Q.E.D.

Consider the functor

$$\text{Sol}_{\hbar, \text{loc}}(\cdot): \text{D}^{\text{b}}(\mathcal{D}_X^{\hbar, \text{loc}})^{\text{op}} \rightarrow \text{D}^{\text{b}}(\mathbb{C}_X^{\hbar, \text{loc}}), \quad \mathcal{M} \mapsto \text{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar, \text{loc}}}(\mathcal{M}, \mathcal{O}_X^{\hbar, \text{loc}}).$$

Proposition 7.8. *Let $\mathcal{M} \in \text{D}_{\text{coh}}^{\text{b}}(\mathcal{D}_X^{\hbar, \text{loc}})$. Then*

$$\text{SS}(\text{Sol}_{\hbar, \text{loc}}(\mathcal{M})) \subset \text{char}_{\hbar, \text{loc}}(\mathcal{M}).$$

Proof. By dévissage, we can assume that $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar, \text{loc}})$. Moreover, since the problem is local, we may assume that \mathcal{M} admits a lattice \mathcal{L} .

One has the isomorphism $\text{Sol}_{\hbar, \text{loc}}(\mathcal{M}) \simeq \text{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{L}, \mathcal{O}_X^{\hbar, \text{loc}})$ by extension of scalars. Taking a local resolution of \mathcal{L} by free \mathcal{D}_X^{\hbar} -modules of finite type, we deduce that $\text{Sol}_{\hbar, \text{loc}}(\mathcal{M}) \simeq F^{\text{loc}}$ for $F = \text{Sol}_{\hbar}(\mathcal{L})$. The statement follows by (7.2) and Corollary 3.13. Q.E.D.

One says that \mathcal{M} is holonomic if its characteristic variety is isotropic.

Proposition 7.9. *Let $\mathcal{M} \in \text{D}_{\text{hol}}^{\text{b}}(\mathcal{D}_X^{\hbar, \text{loc}})$. Then $\text{Sol}_{\hbar, \text{loc}}(\mathcal{M}) \in \text{D}_{\mathbb{C}\text{-c}}^{\text{b}}(\mathbb{C}_X^{\hbar, \text{loc}})$.*

Proof. By the same arguments and with the same notations as in the proof of Proposition 7.8, we reduce to the case $\text{Sol}_{\hbar, \text{loc}}(\mathcal{M}) \simeq F^{\text{loc}}$, for $F = \text{Sol}_{\hbar}(\mathcal{L})$ and \mathcal{L} a lattice of $\mathcal{M} \in \text{Mod}_{\text{hol}}(\mathcal{D}_X^{\hbar, \text{loc}})$. Hence \mathcal{L} is a holonomic \mathcal{D}_X^{\hbar} -module, and $F \in \text{D}_{\mathbb{C}\text{-c}}^{\text{b}}(\mathbb{C}_X^{\hbar})$. Q.E.D.

Remark 7.10. In general the functor

$$\text{Sol}_{\hbar, \text{loc}}: \text{D}_{\text{hol}}^{\text{b}}(\mathcal{D}_X^{\hbar, \text{loc}})^{\text{op}} \rightarrow \text{D}_{\mathbb{C}\text{-c}}^{\text{b}}(\mathbb{C}_X^{\hbar, \text{loc}})$$

is not locally essentially surjective. In fact, consider the quasi-commutative diagram of categories

$$\begin{array}{ccc} \mathbf{D}_{\text{hol}}^b(\mathscr{D}_X^{\hbar})^{\text{op}} & \xrightarrow{\text{Sol}_{\hbar}} & \mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar}) \\ (\cdot)^{\text{loc}} \downarrow & & (\cdot)^{\text{loc}} \downarrow \\ \mathbf{D}_{\text{hol}}^b(\mathscr{D}_X^{\hbar, \text{loc}})^{\text{op}} & \xrightarrow{\text{Sol}_{\hbar, \text{loc}}} & \mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar, \text{loc}}). \end{array}$$

By the local existence of lattices the left vertical arrow is locally essentially surjective. If $\text{Sol}_{\hbar, \text{loc}}$ were also locally essentially surjective, so should be the right vertical arrow. The following example shows that it is not the case.

Example 7.11. Let $X = \mathbb{C}$, $U = X \setminus \{0\}$ and denote by $j: U \hookrightarrow X$ the embedding. Set $F = \text{R}j_*L$, where L is the local system on U with stalk $\mathbb{C}^{\hbar, \text{loc}}$ and monodromy \hbar . There is no $F_0 \in \mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar})$ such that $F \simeq (F_0)^{\text{loc}}$.

One can interpret this phenomenon by remarking that $\mathbf{D}_{\text{hol}}^b(\mathscr{D}_X^{\hbar, \text{loc}})$ is equivalent to the localization of the category $\mathbf{D}_{\text{hol}}^b(\mathscr{D}_X^{\hbar})$ with respect to the morphism \hbar , contrarily to the category $\mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar, \text{loc}})$.

8 Links with deformation quantization

In this last section, we shall briefly explain how the study of deformation quantization algebras on complex symplectic manifolds is related to \mathscr{D}_X^{\hbar} . We follow the terminology of [11].

The cotangent bundle $\mathfrak{X} = T^*X$ to the complex manifold X has a structure of a complex symplectic manifold and is endowed with the \mathbb{C}^{\hbar} -algebra $\widehat{\mathscr{W}}_{\mathfrak{X}}$, a non homogeneous version of the algebra of microdifferential operators. Its subalgebra $\widehat{\mathscr{W}}_{\mathfrak{X}}(0)$ of operators of order at most zero is a deformation quantization algebra. In a system (x, u) of local symplectic coordinates, $\widehat{\mathscr{W}}_{\mathfrak{X}}(0)$ is identified with the star algebra $(\mathcal{O}_{\mathfrak{X}}^{\hbar}, \star)$ in which the star product is given by the Leibniz product:

$$(8.1) \quad f \star g = \sum_{\alpha \in \mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!} (\partial_u^{\alpha} f)(\partial_x^{\alpha} g), \quad \text{for } f, g \in \mathcal{O}_{\mathfrak{X}}.$$

In this section we will set for short $\mathscr{A} := \widehat{\mathscr{W}}_{\mathfrak{X}}(0)$, so that $\mathscr{A}^{\text{loc}} \simeq \widehat{\mathscr{W}}_{\mathfrak{X}}$. Note that \mathscr{A} satisfies Assumption 1.8.

Let us identify X with the zero section of the cotangent bundle \mathfrak{X} . Recall that X is a local model for any smooth Lagrangian submanifold of \mathfrak{X} , and that \mathcal{O}_X^h is a local model of any simple \mathcal{A} -module along X . As \mathcal{O}_X^h has both a \mathcal{D}_X^h -module and an $\mathcal{A}|_X$ -module structure, there are morphisms of \mathbb{C}^h -algebras

$$(8.2) \quad \mathcal{D}_X^h \rightarrow \mathcal{E}nd_{\mathbb{C}^h}(\mathcal{O}_X^h) \leftarrow \mathcal{A}|_X.$$

Lemma 8.1. *The morphisms in (8.2) are injective and induce an embedding $\mathcal{A}|_X \hookrightarrow \mathcal{D}_X^h$.*

Proof. Since the problem is local, we may choose a local symplectic coordinate system (x, u) on \mathfrak{X} such that $X = \{u = 0\}$. Then $\mathcal{A}|_X$ is identified with $\mathcal{O}_{\mathfrak{X}}^h|_X$. As the action of u_i on \mathcal{O}_X^h is given by $\hbar \partial_{x_i}$, the morphism $\mathcal{A}|_X \rightarrow \mathcal{E}nd_{\mathbb{C}^h}(\mathcal{O}_X^h)$ factors through \mathcal{D}_X^h , and the induced morphism $\mathcal{A}|_X \rightarrow \mathcal{D}_X^h$ is described by

$$(8.3) \quad \sum_{i \in \mathbb{N}} f_i(x, u) \hbar^i \mapsto \sum_{j \in \mathbb{N}} \left(\sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq j} \partial_u^\alpha f_{j-|\alpha|}(x, 0) \partial_x^\alpha \right) \hbar^j,$$

which is clearly injective. Q.E.D.

Consider the following subsheaves of \mathcal{D}_X^h

$$\mathcal{D}_X^{h,m} = \prod_{i \geq 0} (F_{i+m} \mathcal{D}_X) \hbar^i, \quad \mathcal{D}_X^{h,f} = \bigcup_{m \geq 0} \mathcal{D}_X^{h,m}.$$

Note that $\mathcal{D}_X^{h,0}$ and $\mathcal{D}_X^{h,f}$ are subalgebras of \mathcal{D}_X^h , that $\mathcal{D}_X^{h,0}$ is \hbar -complete while $\mathcal{D}_X^{h,f}$ is not and that $\mathcal{D}_X^{h,0,\text{loc}} \simeq \mathcal{D}_X^{h,f,\text{loc}}$. By (8.3), the image of $\mathcal{A}|_X$ in \mathcal{D}_X^h is contained in $\mathcal{D}_X^{h,0}$. (The ring $\mathcal{D}_X^{h,0}$ should be compared with the ring $\mathcal{R}_{X \times \mathbb{C}}$ of [14].)

Remark 8.2. More precisely, denote by $\mathcal{O}_{\mathfrak{X}}^{\hat{h}}|_X \simeq (\mathcal{O}_{\mathfrak{X}}^{\hat{h}}|_X)^h$ the formal restriction of $\mathcal{O}_{\mathfrak{X}}^h$ along the submanifold X . Then the star product in (8.1) extends to this sheaf, and (8.3) induces an isomorphism $(\mathcal{O}_{\mathfrak{X}}^{\hat{h}}|_X, \star) \simeq \mathcal{D}_X^{h,0}$.

Summarizing, one has the compatible embeddings of algebras

$$\begin{array}{ccccccc} \mathcal{A}^{\text{loc}}|_X & \hookrightarrow & \mathcal{D}_X^{h,0,\text{loc}} & \xrightarrow{\sim} & \mathcal{D}_X^{h,f,\text{loc}} & \hookrightarrow & \mathcal{D}_X^{h,\text{loc}} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathcal{A}|_X & \hookrightarrow & \mathcal{D}_X^{h,0} & \hookrightarrow & \mathcal{D}_X^{h,f} & \hookrightarrow & \mathcal{D}_X^h \end{array}$$

One has

$$\mathrm{gr}_h \mathcal{A}|_X \simeq \mathcal{O}_{\mathfrak{X}}|_X, \quad \mathrm{gr}_h \mathcal{D}_X^{\hbar,0} \simeq \mathcal{O}_{\hat{\mathfrak{X}}}|_X, \quad \mathrm{gr}_h \mathcal{D}_X^{\hbar,f} \simeq \mathrm{gr}_h \mathcal{D}_X^{\hbar} \simeq \mathcal{D}_X.$$

Proposition 8.3. (i) *The algebra $\mathcal{D}_X^{\hbar,0}$ is faithfully flat over $\mathcal{A}|_X$.*

(ii) *The algebra $\mathcal{D}_X^{\hbar,\mathrm{loc}}$ is flat over $\mathcal{A}^{\mathrm{loc}}|_X$.*

Proof. (i) follows from Theorem 1.12.

(ii) follows from (i) and the isomorphism $(\mathcal{D}_X^{\hbar,0})^{\mathrm{loc}} \simeq \mathcal{D}_X^{\hbar,\mathrm{loc}}$. Q.E.D.

The next examples show that the scalar extension functor

$$\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X^{\hbar,0}) \rightarrow \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X^{\hbar})$$

is neither exact nor full.

Example 8.4. Let $X = \mathbb{C}^2$ with coordinates (x, y) . Then $\hbar \partial_y$ is injective as an endomorphism of $\mathcal{D}_X^{\hbar,0}/\langle \hbar \partial_x \rangle$ but it is not injective as an endomorphism of $\mathcal{D}_X^{\hbar}/\langle \hbar \partial_x \rangle$, since ∂_x belongs to its kernel. This shows that \mathcal{D}_X^{\hbar} is not flat over $\mathcal{D}_X^{\hbar,0}$.

Example 8.5. This example was communicated to us by Masaki Kashiwara. Let $X = \mathbb{C}$ with coordinate x , and denote by (x, u) the symplectic coordinates on $\mathfrak{X} = T^*\mathbb{C}$. Consider the cyclic \mathcal{A} -modules

$$\mathcal{M} = \mathcal{A}/\langle x - u \rangle, \quad \mathcal{N} = \mathcal{A}/\langle x \rangle,$$

and their images in $\mathrm{Mod}(\mathcal{D}_X^{\hbar})$

$$\mathcal{M}' = \mathcal{D}_X^{\hbar}/\langle x - \hbar \partial_x \rangle, \quad \mathcal{N}' = \mathcal{D}_X^{\hbar}/\langle x \rangle.$$

As their supports in \mathfrak{X} differ, \mathcal{M} and \mathcal{N} are not isomorphic as \mathcal{A} -modules. On the other hand, in \mathcal{D}_X^{\hbar} one has the relation

$$(8.4) \quad x \cdot e^{\hbar \partial_x^2/2} = e^{\hbar \partial_x^2/2} \cdot (x - \hbar \partial_x),$$

and hence an isomorphism $\mathcal{M}' \xrightarrow{\sim} \mathcal{N}'$ given by $[P] \mapsto [P \cdot e^{-\hbar \partial_x^2/2}]$. In fact, one checks that

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})|_X = 0, \quad \mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}', \mathcal{N}') \simeq \mathbb{C}_X^{\hbar}.$$

A Complements on constructible sheaves

Let us review some results, well-known from the specialists (see *e.g.*, [15, Proposition 3.10]), but which are usually stated over a field, and we need to work here over the ring \mathbb{C}^h .

Let \mathbb{K} be a commutative unital Noetherian ring of finite global dimension. Assume that \mathbb{K} is syzygic, i.e. that any finitely generated \mathbb{K} -module admits a finite projective resolution by finite free modules. (For our purposes we will either have $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{C}^h$).

Let X be a real analytic manifold. Denote by $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$ the abelian category of \mathbb{R} -constructible sheaves on X and by $\text{D}_{\mathbb{R}\text{-c}}^b(\mathbb{K}_X)$ the bounded derived category of sheaves of \mathbb{K} -modules with \mathbb{R} -constructible cohomology.

For the next two lemmas we recall some notations and results of [4, 7]. We consider a simplicial complex $\mathbf{S} = (S, \Delta)$, with set of vertices S and set of simplices Δ . We let $|\mathbf{S}|$ be the realization of \mathbf{S} . Thus $|\mathbf{S}|$ is the disjoint union of the realizations $|\sigma|$ of the simplices. For a simplex $\sigma \in \Delta$, the open set $U(\sigma)$ is defined in [7, (8.1.3)]. A sheaf F of \mathbb{K} -modules on $|\mathbf{S}|$ is said weakly \mathbf{S} -constructible if, $\forall \sigma \in \Delta$, $F|_{|\sigma|}$ is constant. An object $F \in \text{D}^b(\mathbb{K}_{|\mathbf{S}|})$ is said weakly \mathbf{S} -constructible if its cohomology sheaves are so. If moreover, all stalks F_x are perfect complexes, F is said \mathbf{S} -constructible. By [7, Proposition 8.1.4] we have isomorphisms, for a weakly \mathbf{S} -constructible sheaf F and for any $\sigma \in \Delta$ and $x \in |\sigma|$:

$$(A.1) \quad \Gamma(U(\sigma); F) \xrightarrow{\sim} \Gamma(|\sigma|; F) \xrightarrow{\sim} F_x,$$

$$(A.2) \quad H^j(U(\sigma); F) = H^j(|\sigma|; F) = 0, \quad \text{for } j \neq 0.$$

It follows that, for a weakly \mathbf{S} -constructible $F \in \text{D}^b(\mathbb{K}_{|\mathbf{S}|})$, the natural morphisms of complexes of \mathbb{K} -modules

$$(A.3) \quad \Gamma(U(\sigma); F) \rightarrow \Gamma(|\sigma|; F) \rightarrow F_x$$

are quasi-isomorphisms.

For $U \subset X$ an open subset, we denote by $\mathbb{K}_U := (\mathbb{K}_X)_U$ the extension by 0 of the constant sheaf on U .

Proposition A.1. *Let $F \in \text{D}_{\mathbb{R}\text{-c}}^b(\mathbb{K}_X)$. Then*

(i) *F is isomorphic to a complex*

$$0 \rightarrow \bigoplus_{i_a \in I_a} \mathbb{K}_{U_{a,i_a}} \rightarrow \cdots \rightarrow \bigoplus_{i_b \in I_b} \mathbb{K}_{U_{b,i_b}} \rightarrow 0,$$

where the $\{U_{k,i_k}\}_{k,i_k}$'s are locally finite families of relatively compact subanalytic open subsets of X .

(ii) F is isomorphic to a complex

$$0 \rightarrow \bigoplus_{i_a \in I_a} \Gamma_{V_{a,i_a}} \mathbb{K}_X \rightarrow \cdots \rightarrow \bigoplus_{i_b \in I_b} \Gamma_{V_{b,i_b}} \mathbb{K}_X \rightarrow 0,$$

where the $\{V_{k,i_k}\}_{k,i_k}$'s are locally finite families of relatively compact subanalytic open subsets of X .

Proof. (i) By the triangulation theorem for subanalytic sets (see for example [7, Proposition 8.2.5]) we may assume that F is an \mathbf{S} -constructible object in $\mathbf{D}^b(\mathbb{K}_{|\mathbf{S}|})$ for some simplicial complex $\mathbf{S} = (S, \Delta)$. For i an integer, let $\Delta_i \subset \Delta$ be the subset of simplices of dimension $\leq i$ and set $\mathbf{S}_i = (S, \Delta_i)$. We denote by $\mathbf{K}^b(\mathbb{K})$ (resp. $\mathbf{K}^b(\mathbb{K}_{|\mathbf{S}|})$) the category of bounded complexes of \mathbb{K} -modules (resp. sheaves of \mathbb{K} -modules on $|\mathbf{S}|$) with morphisms up to homotopy. We shall prove by induction on i that there exists a morphism $u_i: G_i \rightarrow F$ in $\mathbf{K}^b(\mathbb{K}_{|\mathbf{S}|})$ such that:

- (a) the G_i^k are finite direct sums of $\mathbb{K}_{U(\sigma_\alpha)}$'s for some $\sigma_\alpha \in \Delta_i$,
- (b) $u_i|_{|\mathbf{S}_i|}: G_i|_{|\mathbf{S}_i|} \rightarrow F|_{|\mathbf{S}_i|}$ is a quasi-isomorphism.

The desired result is obtained for i equal to the dimension of X .

(i)-(1) For $i = 0$ we consider $F|_{|\mathbf{S}_0|} \simeq \bigoplus_{\sigma \in \Delta_0} F_\sigma$. The complexes $\Gamma(U(\sigma); F)$, $\sigma \in \Delta_0$, have finite bounded cohomology by the quasi-isomorphisms (A.3). Hence we may choose bounded complexes of finite free \mathbb{K} -modules, $R_{0,\sigma}$, and morphisms $u_{0,\sigma}: R_{0,\sigma} \rightarrow \Gamma(U(\sigma); F)$ which are quasi-isomorphisms.

We have the natural isomorphism $\Gamma(U(\sigma); F) \simeq a_* \mathcal{H}om_{\mathbf{K}^b(\mathbb{K}_{|\mathbf{S}|})}(\mathbb{K}_{U(\sigma)}, F)$ in $\mathbf{K}^b(\mathbb{K})$, where $a: |\mathbf{S}| \rightarrow \text{pt}$ is the projection and $\mathcal{H}om$ is the internal Hom functor. We deduce the adjunction formula, for $R \in \mathbf{K}^b(\mathbb{K})$, $F \in \mathbf{K}^b(\mathbb{K}_{|\mathbf{S}|})$:

$$(A.4) \quad \text{Hom}_{\mathbf{K}^b(\mathbb{K})}(R, \Gamma(U(\sigma); F)) \simeq \text{Hom}_{\mathbf{K}^b(\mathbb{K}_{|\mathbf{S}|})}(R_{U(\sigma)}, F).$$

Hence the $u_{0,\sigma}$ induce $u_0: G_0 := \bigoplus_{\sigma \in \Delta_0} (R_{0,\sigma})_{U(\sigma)} \rightarrow F$. By (A.3) $(u_0)_x$ is a quasi-isomorphism for all $x \in |\mathbf{S}_0|$, so that $u_0|_{|\mathbf{S}_0|}$ also is a quasi-isomorphism, as required.

(i)-(2) We assume that u_i is built and let $H_i = M(u_i)[-1]$ be the mapping cone of u_i , shifted by -1 . By the distinguished triangle in $\mathbf{K}^b(\mathbb{K}_{|\mathbf{S}|})$

$$(A.5) \quad H_i \xrightarrow{v_i} G_i \xrightarrow{u_i} F \xrightarrow{+1}$$

$H_i|_{|\mathbf{S}_i|}$ is quasi-isomorphic to 0. Hence $\bigoplus_{\sigma \in \Delta_{i+1} \setminus \Delta_i} (H_i)_{|\sigma|} \rightarrow H_i|_{|\mathbf{S}_{i+1}|}$ is a quasi-isomorphism. As above we choose quasi-isomorphisms $u_{i,\sigma}: R_{i+1,\sigma} \rightarrow \Gamma(U(\sigma); H_i)$, $\sigma \in \Delta_{i+1} \setminus \Delta_i$, where the $R_{i+1,\sigma}$ are bounded complexes of finite free \mathbb{K} -modules. By (A.4) again the $u_{i,\sigma}$ induce a morphism in $\mathbf{K}^b(\mathbb{K}_{|\mathbf{S}|})$

$$u'_{i+1}: G'_{i+1} := \bigoplus_{\sigma \in \Delta_{i+1} \setminus \Delta_i} (R_{i+1,\sigma})_{U(\sigma)} \rightarrow H_i.$$

For $x \in |\mathbf{S}_{i+1}| \setminus |\mathbf{S}_i|$, $(u'_{i+1})_x$ is a quasi-isomorphism by (A.3), and, for $x \in |\mathbf{S}_i|$, this is trivially true. Hence $u'_{i+1}|_{|\mathbf{S}_{i+1}|}$ is a quasi-isomorphism.

Now we let H_{i+1} and G_{i+1} be the mapping cones of u'_{i+1} and $v_i \circ u'_{i+1}$, respectively. We have distinguished triangles in $\mathbf{K}^b(\mathbb{K}_{|\mathbf{S}|})$

$$(A.6) \quad G'_{i+1} \xrightarrow{u'_{i+1}} H_i \rightarrow H_{i+1} \xrightarrow{+1}, \quad G'_{i+1} \xrightarrow{v_i \circ u'_{i+1}} G_i \rightarrow G_{i+1} \xrightarrow{+1}.$$

By the construction of the mapping cone, the definition of G'_{i+1} and the induction hypothesis, G_{i+1} satisfies property (a) above. The octahedral axiom applied to triangles (A.5) and (A.6) gives a morphism $u_{i+1}: G_{i+1} \rightarrow F$ and a distinguished triangle $H_{i+1} \rightarrow G_{i+1} \xrightarrow{u_{i+1}} F \xrightarrow{+1}$. By construction $H_{i+1}|_{|\mathbf{S}_{i+1}|}$ is quasi-isomorphic to 0 so that u_{i+1} satisfies property (b) above.

(ii) Consider the duality functor $D'_{\mathbb{K}}(\cdot) = R\mathcal{H}om_{\mathbb{K}_X}(\cdot, \mathbb{K}_X)$. Set $G = D'_{\mathbb{K}}(F)$, and represent it by a bounded complex as in (i). Since U_{k,i_k} corresponds to an open subset of the form $U(\sigma)$ in $|\mathbf{S}|$, the sheaves $\mathbb{K}_{U_{k,i_k}}$ are acyclic for the functor $D'_{\mathbb{K}}$. Hence $F \simeq D'_{\mathbb{K}}(G)$ can be represented as claimed.

Q.E.D.

Lemma A.2. *Let $F \rightarrow G \rightarrow 0$ be an exact sequence in $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$. Then for any relatively compact subanalytic open subset $U \subset X$, there exists a finite covering $U = \bigcup_{i \in I} U_i$ by subanalytic open subsets such that, for each $i \in I$, the morphism $F(U_i) \rightarrow G(U_i)$ is surjective.*

Proof. As in the proof of Proposition A.1 we may assume that F and G are constructible sheaves on the realization of some finite simplicial complex (S, Δ) . For $\sigma \in \Delta$ the morphism $\Gamma(U(\sigma); F) \rightarrow \Gamma(U(\sigma); G)$ is surjective, by (A.1). Since $|\mathbf{S}|$ is the finite union of the $U(\sigma)$ this proves the lemma.
Q.E.D.

B Complements on subanalytic sheaves

We review here some well-known results (see [9, Chapter 7] and [13]) but which are usually stated over a field, and we need to work here over the ring \mathbb{C}^h .

Let \mathbb{K} be a commutative unital Noetherian ring of finite global dimension (for our purposes we will either have $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{C}^h$). Let X be a real analytic manifold, and consider the natural morphism $\rho: X \rightarrow X_{\text{sa}}$ to the associated subanalytic site.

Lemma B.1. *The functor $\rho_*: \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X) \rightarrow \text{Mod}(\mathbb{K}_{X_{\text{sa}}})$ is exact and $\rho^{-1}\rho_*$ is isomorphic to the canonical functor $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X) \rightarrow \text{Mod}(\mathbb{K}_X)$.*

Proof. Being a direct image functor, ρ_* is left exact. It is right exact thanks to Lemma A.2. The composition $\rho^{-1}\rho_*$ is isomorphic to the identity on $\text{Mod}(\mathbb{K}_X)$ since the open sets of the site X_{sa} give a basis of the topology of X . Q.E.D.

In the sequel, we denote by $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\text{sa}}})$ the image by the functor ρ_* of $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$ in $\text{Mod}(\mathbb{K}_{X_{\text{sa}}})$. Hence ρ_* induces an equivalence of categories $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X) \simeq \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\text{sa}}})$. We also denote by $\text{D}_{\mathbb{R}\text{-c}}^b(\mathbb{K}_{X_{\text{sa}}})$ the full triangulated subcategory of $\text{D}^b(\mathbb{K}_{X_{\text{sa}}})$ consisting of objects with cohomology in $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\text{sa}}})$.

Corollary B.2. *The subcategory $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\text{sa}}})$ of $\text{Mod}(\mathbb{K}_{X_{\text{sa}}})$ is thick.*

Proof. Since ρ_* is fully faithful and exact, $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\text{sa}}})$ is stable by kernel and cokernel. It remains to see that, for $F, G \in \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$

$$\text{Ext}_{\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)}^1(F, G) \simeq \text{Ext}_{\text{Mod}(\mathbb{K}_{X_{\text{sa}}})}^1(\rho_*F, \rho_*G).$$

By [4] we know that the first Ext^1 may as well be computed in $\text{Mod}(\mathbb{K}_X)$. We see that the functors ρ^{-1} and $R\rho_*$ between $\text{D}^b(\mathbb{K}_X)$ and $\text{D}^b(\mathbb{K}_{X_{\text{sa}}})$ are adjoint, and moreover $\rho^{-1}R\rho_* \simeq \text{id}$. Thus, for $F', G' \in \text{D}^b(\mathbb{K}_X)$ we have

$$\text{Hom}_{\text{D}^b(\mathbb{K}_{X_{\text{sa}}})}(R\rho_*F', R\rho_*G') \simeq \text{Hom}_{\text{D}^b(\mathbb{K}_X)}(F', G'),$$

and this gives the result. Q.E.D.

This corollary gives the equivalence $\text{D}_{\mathbb{R}\text{-c}}^b(\mathbb{K}_X) \simeq \text{D}_{\mathbb{R}\text{-c}}^b(\mathbb{K}_{X_{\text{sa}}})$, both categories being equivalent to $\text{D}^b(\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X))$.

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